1. Choice, Preference, & Utility

(Reference: Chapter 1, Decision Theory, Tomasz Strzalecki)

X is a set of alternatives that the agent is choosing between. There are 3 languages to describe choice:

- **Choice function (or correspondence)**: Observe which element (or elements) the agent chooses from every nonempty subset $A \subseteq X$. $c(A)$ denotes that element.
- **Preference relation:** Agent ranks each pair of items $x, y \in X$: $x \succ y$, $y \succ x$, or $x \sim y$
- **Utility function:** $u : X \to \mathbb{R}$

Revealed Preference Theory: choice function $\rightarrow$ preference relation

What are the conditions on choice functions such that they can equivalently be represented by maximization of a preference relation?

Choice Function

$M(X) = 2^X \backslash \{\emptyset\}$: the set of all potential menus (all subsets of X not including the empty set)

If c is a **choice function**:

- If X is **finite**:
  - $c : M(X) \to X$ such that $c(A) \in A$ for all $A \in M(X)$
- If X is **infinite**:
  - Let $M \subset M(X)$ be a collection of nonempty finite subsets of X that contains all pairs and triples
    - Our proof of Propositions 1.1 and 1.2 relies on $\succsim$ being complete and transitive, which requires the observation of choice in all sets with pairs and triples, but nothing more
    - M is an infinite set, but includes only finite subsets of X
  - $c : M \to X$ such that $c(A) \in A$ for all $A \in M$
- In both cases, we define the domain of c to not include the empty set, so we always choose something ($c(A)$ is nonempty)
- In both cases, we define the domain of c to only contain finite menus – otherwise, we may not be able to maximize choice from the menu (e.g., you can always choose something on the interval [0,1) but it is not possible to choose the highest value, so $c_{\succsim}$ would be empty)

If c is a **choice correspondence**, $c(A)$ can contain multiple elements (c maps sets to sets):

- If X is **finite**: $c : M(X) \to M(X)$ such that $c(A) \subseteq A$ for all $A \in M(X)$
- If X is **infinite**: $c : M \to M(X)$ such that $c(A) \subseteq A$ for all $A \in M$

Preference Relation

Weak preference: $x \succsim y \;\Leftrightarrow\; (x,y) \in X \times X \ni$ "x is weakly preferred to y" $\quad (\succsim \;\subseteq\; X \times X)$

- Relation notation: $x \succsim y \;\Leftrightarrow\; (x,y) \in \,\succsim$
- **Complete:** $x \succsim y \;\vee\; y \succsim x \quad \forall x,y \in X \qquad (\Rightarrow \; x \succsim x)$
- **Transitive:** $x \succsim y \;\wedge\; y \succsim z \;\Rightarrow\; x \succsim z \;\; \forall x,y,z \in X$
- **Antisymmetric:** $x \succsim y \;\wedge\; y \succsim x \;\Rightarrow\; x = y \;\; \forall x,y \in X$ (not always assumed; rules out indifference)
- Reflexive: $x \succsim x \;\; \forall x \in X$ (implied by completeness)

Strict preference: $x \succ y \;\Leftrightarrow\; x \succsim y \;\wedge\; \neg(y \succsim x)$

Indifference: $x \sim y \;\Leftrightarrow\; x \succsim y \;\wedge\; y \succsim x$

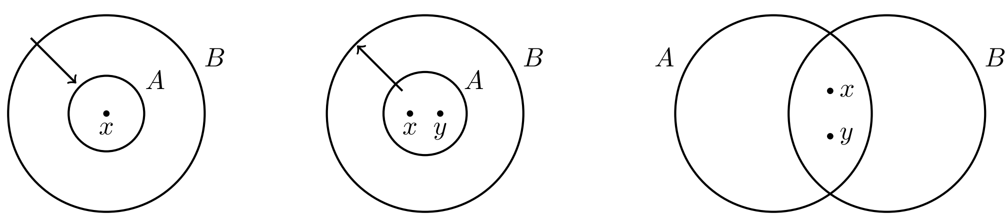
We define $c_{\succsim}$ as a choice function (or correspondence) induced by the maximization of $\succsim$:

- **Choice function:** $c_{\succsim}(A) = x$ iff $x \in A$ and $x \succsim y$ for all $y \in A$
- **Choice correspondence:** $c_{\succsim}(A) = \{x \in A \mid x \succsim y \;\; \forall y \in A\}$ $\quad$ ($c_{\succsim}(A)$ is a set)
- $c_{\succsim}$ is the only c we have defined (they have the same domain and range)

Sen's $\boldsymbol{\alpha}$ (contraction): $x \in A \subseteq B \;\wedge\; x \in c(B) \;\Rightarrow\; x \in c(A) \qquad (x = c(\cdot)$ for choice function)

Sen's $\boldsymbol{\beta}$ (expansion): $x, y \in A \subseteq B \;\wedge\; x \in c(A) \;\wedge\; y \in c(A) \;\wedge\; y \in c(B) \;\Rightarrow\; x \in c(B)$

WARP (Venn): $x, y \in A \cap B \;\wedge\; x \in c(A) \;\wedge\; y \in c(B) \;\Rightarrow\; x \in c(B) \;\wedge\; y \in c(A)$



Sen's α $\qquad$ Sen's β $\qquad$ WARP

Without Indifference

- We assume $\succsim$ is <u>complete, transitive, and antisymmetric</u> $\;\Rightarrow\; \exists! x \ni x \succsim y \;\; \forall y \in A$
- c and $c_{\succsim}$ are choice <u>functions</u>, $c(A)$ and $c_{\succsim}(A)$ are <u>single-valued</u>

Proposition 1.1 (without indifference) The following are equivalent:

(i) A choice function c satisfies Sen's α condition

(ii) There exists a complete, transitive, and antisymmetric $\succsim$ such that $c = c_{\succsim}$

With Indifference

- We assume $\succsim$ is complete and transitive $\Rightarrow$ $\exists x \ni x \succsim y \;\; \forall y \in A$
- c and $c_{\succsim}$ are choice correspondences, $c(A)$ and $c_{\succsim}(A)$ can be single-valued or multi-valued sets

Proposition 1.2 (with indifference) The following are equivalent:

(i) A choice correspondence c satisfies Sen's α and β conditions

(ii) A choice correspondence c satisfies WARP

(iii) There exists a complete and transitive $\succsim$ such that $c = c_{\succsim}$

Utility Representation Theory: preference relation $\rightarrow$ utility function

What are the conditions under which a preference relation can be represented by a utility function?

$\succsim$ is represented by $u : X \rightarrow \mathbb{R}$ iff $\forall x, y \in X$: $\quad x \succsim y \;\; \Leftrightarrow \;\; u(x) \geq u(y)$

- If $\succsim$ has a utility representation, then the sets $(X, \succsim)$ and $(\mathbb{R}, \geq)$ look the same from the point of view of order theory (we can use what we know about the ordering of real numbers)
- Implies that $x \succ y \;\; \Leftrightarrow \;\; u(x) > u(y)$

Proposition 1.3 Suppose X is finite. The following statements are equivalent:

(i) $\succsim$ is complete and transitive

(ii) $\succsim$ is represented by some utility function $u : X \rightarrow \mathbb{R}$

$\succsim$ is continuous if for any sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ with $x_n \rightarrow x$ and $y_n \rightarrow y$, and $x_n \succsim y_n \;\; \forall n$, we have $x \succsim y$

- $\Leftrightarrow$ for all x, the upper and lower contour sets are both closed

Proposition 1.4 Suppose $X = \mathbb{R}^L_+$. The following statements are equivalent:

(i) $\succsim$ is complete, transitive, and continuous

(ii) $\succsim$ is represented by some continuous utility function $u : X \rightarrow \mathbb{R}$

The utility function $u(\cdot)$ that represents $\succsim$ is not unique; any (strictly) increasing transformation of $u(\cdot)$ also represents $\succsim$ (represents the same order of preferences). Therefore, not all utility functions that represent $\succsim$ are continuous.

2a. Consumer Theory: Choice-Based Demand

(Reference: Chapter 2, Microeconomic Theory, Mas-Colell, Whinston, & Green)

We formulate a theory of demand based on choice by looking more specifically at consumption sets X, and choice of bundle x from budget sets $B_{p,w} \subset X$

Consumption & Budget Sets

Commodity space: $\mathbb{R}^L$ for some finite L number of goods

Bundle: Vector (point in the commodity space) with quantities of each commodity $\underset{L\times 1}{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_L \end{bmatrix}$

Consumption set: $X \subseteq \mathbb{R}^L$ whose elements are the commodity bundles the individual can possibly consume

- Default: $X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L \mid x_\ell \geq 0 \text{ for } \ell = 1, ..., L\}$, the set of all non-negative bundles
- Note: $\mathbb{R}^L_+$ is a convex set

Price: Vector with unit price of each commodity $\underset{L\times 1}{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} \in \mathbb{R}^L$ (typically assume $p \gg 0$)

- We assume that markets are complete (all commodities are traded at publicly available prices) and that consumers are price-taking

Walrasian budget set: $\boldsymbol{B_{p,w}} = \{x \in \mathbb{R}^L_+ \mid p \bullet x \leq w\}$

- Typically assume $w > 0$
- Note: $B_{p,w}$ is a convex and compact set (closed and bounded)

$p \bullet x \; = \; p^T \cdot x$ (take the inner product of two $L \times 1$ vectors to get a scalar)

Demand Function

The **Walrasian demand function** $\boldsymbol{x(p,w)}$ designates a chosen bundle for each price-wealth pair (p, w)

- $x(p, w) \in B_{p,w}$
- We assume the demand function is continuous and differentiable for simplicity
- We assume $x(p, w)$ is single-valued – if $x(p, w)$ can be multi-valued (the individual might choose any of a number of bundles), we call it the demand correspondence

Price effect (total effect): The effect of a change in p_k on demand for good ℓ: $\dfrac{\partial x_\ell(p,w)}{\partial p_k}$

Elasticity w.r.t. price (price effect as %): $\varepsilon_{\ell k}(p,w) = \dfrac{\partial x_\ell(p,w)}{\partial p_k} \cdot \dfrac{p_k}{x_\ell(p,w)}$

Wealth effects (income effect): The effect of a change in w on demand for good ℓ: $\dfrac{\partial x_\ell(p,w)}{\partial w}$

- ≥ 0 for normal goods
- < 0 for inferior goods

Elasticity w.r.t. wealth (wealth effect as %): $\varepsilon_{\ell w}(p,w) = \dfrac{\partial x_\ell(p,w)}{\partial w} \cdot \dfrac{w}{x_\ell(p,w)}$

$$\underset{L\times L}{D_p x(p,w)} = \begin{bmatrix} \dfrac{\partial x_1}{\partial p_1} & \cdots & \dfrac{\partial x_1}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_L}{\partial p_1} & \cdots & \dfrac{\partial x_L}{\partial p_L} \end{bmatrix} \qquad \underset{L\times 1}{D_w x(p,w)} = \begin{bmatrix} \dfrac{\partial x_1}{\partial w} \\ \vdots \\ \dfrac{\partial x_L}{\partial w} \end{bmatrix}$$

Homogeneous of degree zero: $x(\alpha p, \alpha w) = x(p,w) \quad \forall p, w, \alpha > 0$ (changing currency doesn't matter)

Differentiating w.r.t. α: $$\sum_{k=1}^{L} \frac{\partial x_\ell(p,w)}{\partial p_k} \cdot p_k + \frac{\partial x_\ell(p,w)}{\partial w} \cdot w = 0 \text{ for } \ell = 1, ..., L$$

$$D_p x(p,w) \cdot p + D_w x(p,w) \cdot w = \underset{L\times 1}{0}$$

Walras' law: $p \cdot x = w \quad \forall x \in x(p,w), p \gg 0, w > 0$ (always spend all your money)

$$\sum_{\ell=1}^{L} p_\ell \cdot x_\ell(p,w) = w$$

Differentiating w.r.t. p_k: $$\sum_{\ell=1}^{L} p_\ell \cdot \frac{\partial x_\ell(p,w)}{\partial p_k} + x_k(p,w) = 0 \text{ for } k = 1, ..., L$$

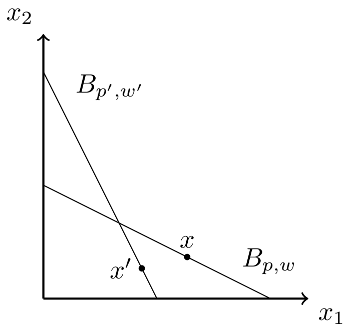
Differentiating w.r.t. p: $$p \cdot D_p x(p,w) + x(p,w)^T = \underset{1\times L}{0^T}$$

Differentiating w.r.t. w: $$\sum_{\ell=1}^{L} p_\ell \cdot \frac{\partial x_\ell(p,w)}{\partial w} = 1$$

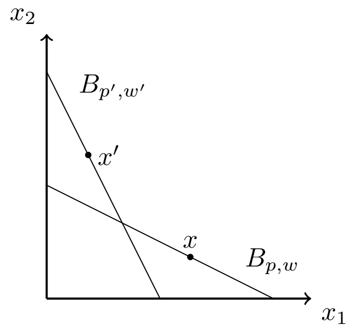
$$p \cdot D_w x(p,w) = 1$$

WARP for demand functions: $p \cdot x' \leq w \ \wedge \ x' \neq x \ \Rightarrow \ p' \cdot x > w'$

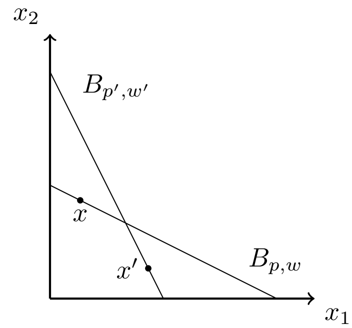
- If you could afford x' at old scheme, then you can't afford x at new scheme
- This property extends to demand correspondences: because we assume Walras' law, x must be on its budget line (hyperplane) – thus the only way for WARP to hold is if x and x' are at the intersection of their budget lines (hyperplanes)
- For demand correspondences: $x \in x(p,w) \ \wedge \ x' \in x(p',w') \ \wedge \ p \cdot x' < w \ \Rightarrow \ p' \cdot x > w'$



WARP is satisfied ($P \Rightarrow Q$)



Consistent with WARP ($\neg P$)



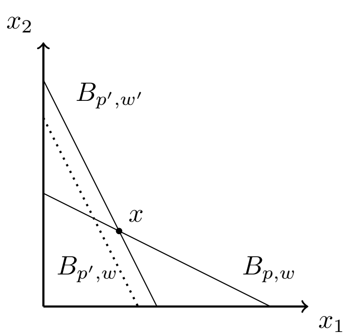
WARP is **not** satisfied ($P \wedge \neg Q$)

Slutsky Compensated Price Changes

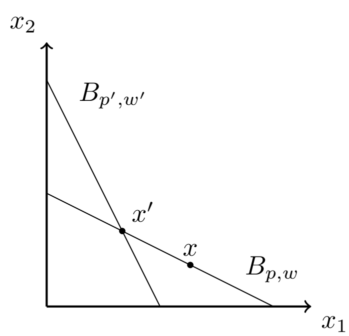
A **Slutsky compensated price change** is accompanied by a wealth change that makes the initial consumption bundle just affordable. Slutsky compensated price changes isolate the substitution effect (how the individual responds to a change in the relative cost of commodities), by removing the income effect (a change in real wealth).

$p \cdot x = w \ \rightarrow \ w' = p' \cdot x$

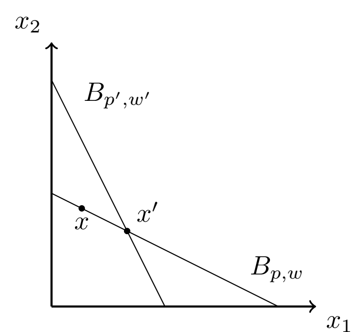
Compensated WARP: $p \cdot x' = w \ \wedge \ x' \neq x \ \Rightarrow \ p' \cdot x > w'$ $\quad (p' \cdot x = w' \ \wedge \ x' \neq x \ \Rightarrow \ p \cdot x' > w)$



Compensated Price Change



CWARP is satisfied



CWARP is **not** satisfied

Proposition 2.2 If Walras' law holds, then WARP $\Leftrightarrow$ Compensated WARP

- WARP holds iff it holds for all compensated price changes
- If WARP is violated, there must be a compensated price change for which it is violated

Compensated Law of Demand (Slutsky): Suppose that the Walrasian demand function x is homogeneous of degree zero and satisfies Walras' law. The following statements are equivalent:

(i) x satisfies WARP

(ii) For any compensated price change from (p, w) to (p', w'), we have $[p' - p] \cdot [x' - x] \leq 0$ (< 0 if $x \neq x'$)

- Follows directly from Proposition 2.2 (multiply out and use CWARP)
- Equivalently $\Delta p \cdot \Delta x \leq 0$ or $dp \cdot dx \leq 0$ – demand and price move in opposite directions (substitution effect is always negative)

Slutsky Matrix

$$\underset{L\times L}{S(p,w)} = \begin{bmatrix} s_{11}(p,w) & \dots & s_{1L}(p,w) \\ \vdots & \ddots & \vdots \\ s_{L1}(p,w) & \dots & s_{LL}(p,w) \end{bmatrix} = D_p x(p,w) + D_w x(p,w)\ x(p,w)^T$$

$$s_{\ell k}(p,w) = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w}\, x_k(p,w)$$

- Substitution Effect = Total Effect + Income Effect
- Effect of a change in p_k on good x_ℓ
- Effect times dp_k gives the size of change in demand ($\frac{\text{commodity units}}{\text{price units}} \times$ price units)
- The first term times dp_k gives us the total effect; the second term times dp_k gives us the income effect, noting that $dw = x_k(p,w)\ dp_k$
- $dp \cdot S(p,w)dp = dp \cdot dx \leq 0$
- "Look! Tim excels at packing and I excel at wrangling extra kangaroos"

Proposition 2.4: If a differentiable Walrasian demand function x satisfies Walras' law, homogeneity of degree zero, and WARP, then at any (p, w), $S(p, w)$ satisfies $v \cdot S(p,w)v \leq 0\ \ \forall v \in \mathbb{R}^L$

- $S(p, w)$ is negative semidefinite by definition
- Diagonal values $s_{\ell\ell}(p,w) \leq 0$ – the substitution effect for a good w.r.t. its own price is nonpositive

Proposition 2.5 If a differentiable Walrasian demand function x satisfies Walras' law and homogeneity of degree zero, then at any (p, w), $S(p, w)$ satisfies $p \cdot S(p, w) = 0$ and $S(p, w)p = 0$

- $S(p, w)$ is singular (i.e., has rank less than L) – one of the rows or columns is a linear combination of the others, so the matrix is not invertible and its determinant is zero
- Thus $S(p, w)$ is negative semidefinite but not negative definite

Under Walras' law, homogeneity of degree zero, and WARP, $S(p, w)$ is symmetric for $L = 2$ but not necessarily symmetric for $L > 2$

Having formulated a theory of demand based on choice, we would hope to use Propositions 1.1 or 1.2 to show that this theory of demand based on choice = theory of demand based on preference maximization. However, they are not equivalent, because the family of Walrasian budget sets does not allow us to define choice on all two- and three-element subsets of X. Hicks 1956 shows a case in $L = 3$ where choices from 3 budget sets are pairwise consistent with WARP, but violate transitivity (choice from the triplet is not well-defined). In fact, the theory of demand based on preference maximization is stronger in that it implies more restrictions (implies the Slutsky matrix is symmetric for all $L \geq 2$).

2b. Consumer Theory: Preference-Based Demand

(Reference: Chapter 3, Microeconomic Theory, Mas-Colell, Whinston, & Green)

Restrictions on Preferences

We assume throughout:

- $X = \mathbb{R}^L_+$
- $\succsim$ is complete, transitive, and continuous (and thus there is a continuous utility function $u(\cdot)$ that represents $\succsim$)

Desirability Assumptions About $\succsim$

Desirability assumptions capture the idea that larger amounts of commodities are preferred to smaller ones.

Strongly monotone: $y \geq x \ \wedge \ y \neq x \ \Rightarrow \ y \succ x$

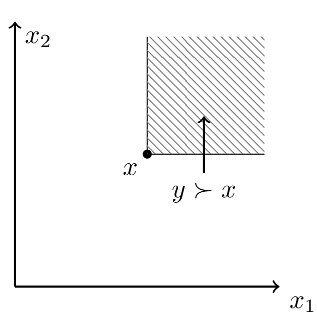
- All commodities are "goods" – cannot be indifferent to an increase in any commodities

Montone: $y \gg x \ \Rightarrow \ y \succ x$

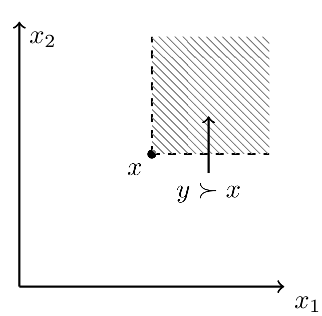
- All commodities are "goods" – may be indifferent to an increase in some (but not all) goods

Locally nonsatiated: $\forall \varepsilon > 0, \quad \exists y \ni ||y - x|| \leq \varepsilon \ \wedge \ y \succ x$

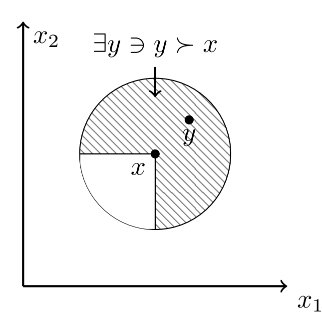
- Some commodities can be "bads", but not all – because we are in $\mathbb{R}^L_+$, we would get stuck at 0
- $||y - x|| = \sqrt{\sum_{\ell=1}^{L} (y_\ell - x_\ell)^2}$



Strong monotonicity



Monotonicity



Local nonsatiation

Strongly montone $\Rightarrow$ monotone $\Rightarrow$ locally nonsatiated

- All three rule out thick indifference curves
- Recall that $y \succ x \;\Leftrightarrow\; u(y) > u(x)\ \forall u(\cdot)$ – monotonicity implies that all equivalent utility functions are increasing

Convexity Assumptions About $\succsim$

Convexity assumptions capture the idea that agents are inclined towards diversification. If agents have convex preferences, they have diminishing marginal rates of substitution between any two goods – from any initial consumption bundle, it takes increasingly larger amounts of one commodity to compensate for successive unit losses of another.

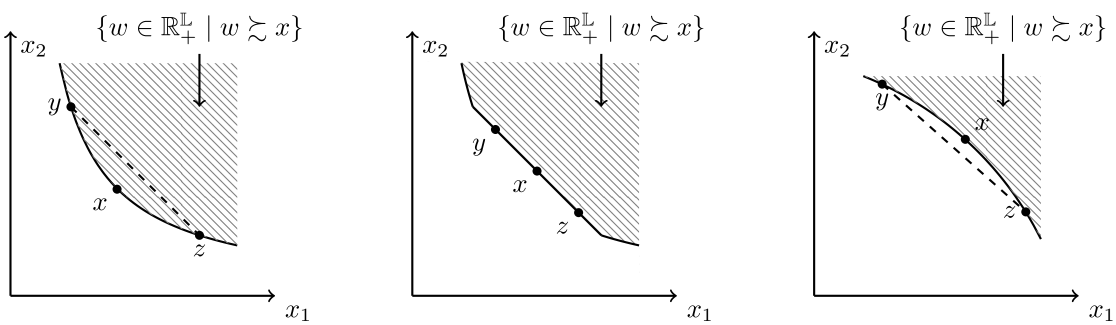
Upper contour set: The set of all bundles that are at least as good as x $\{y \in X \mid y \succsim x\}$

Indifference set: The set of all bundles that are indifferent to x $\{y \in X \mid y \sim x\}$

Lower contour set: The set of all bundles that x is at least as good as $\{y \in X \mid x \succsim y\}$

Convex (upper countour set): $y \succsim x \;\wedge\; z \succsim x \;\Rightarrow\; \alpha y + (1-\alpha)z \succsim x\ \forall \alpha \in [0,1]$

Strictly convex (upper contour set): $y \succsim x \;\wedge\; z \succsim x \;\wedge\; y \neq z \;\Rightarrow\; \alpha y + (1-\alpha)z \succ x\ \forall \alpha \in (0,1)$



Strictly convex — Convex but not strictly — Not convex

Convex $\succsim \;\Leftrightarrow\;$ Quasiconcave $u(\cdot)$ $\forall$ equivalent $u(\cdot)$

Strictly convex $\succsim \;\Leftrightarrow\;$ Strictly quasiconcave $u(\cdot)$ $\forall$ equivalent $u(\cdot)$

Additional Assumptions About $\succsim$

It is common to focus on preferences for which it is possible to deduce the consumer's entire preference relation from a single indifference set. Two examples:

Homothetic: $x \sim y \;\Rightarrow\; \alpha x \sim \alpha y\ \forall \alpha \geq 0$

- Assumes monotonicity
- $\Leftrightarrow\; \exists\ u(\cdot)$ that is homogeneous of degree one: $u(\alpha x) = \alpha u(x)\ \forall \alpha > 0$

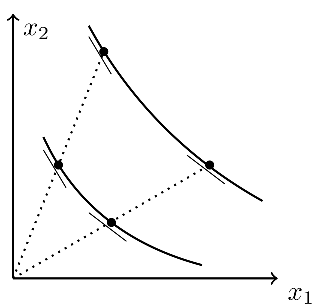
- Constant MRS along rays from the origin (e.g., Cobb-Douglas)

Quasilinear (with respect to commodity 1):

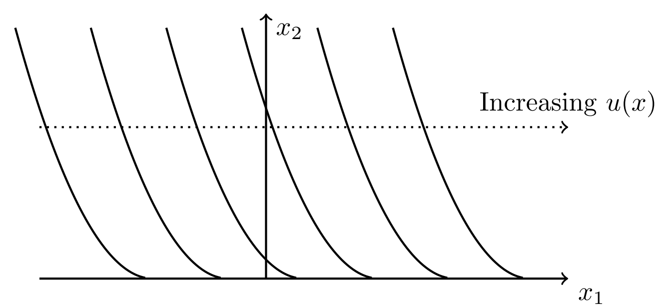
(i) $x \sim y \;\Rightarrow\; (x + \alpha e_1) \sim (y + \alpha e_1), e_1 = (1, 0, ...0) \;\forall \alpha \in \mathbb{R}$

(ii) $x + \alpha e_1 \succ x \;\forall x \;\forall \alpha > 0$ Commodity 1 is a "good"

- $\Leftrightarrow \;\exists\; u(\cdot) \ni u(x) = x_1 + \phi(x_2, \ldots, x_L)$



Homothetic



Quasilinear w.r.t. x_1

Increasing and (strictly) quasiconcave are ordinal properties of $u(\cdot)$, so they will hold for any increasing transformation (any $u(\cdot)$ that represents $\succsim$). However, HOD 1 and quasilinear $u(\cdot)$ are cardinal and not preserved by increasing transformations – we only know that there exists a $u(\cdot)$ with such form.

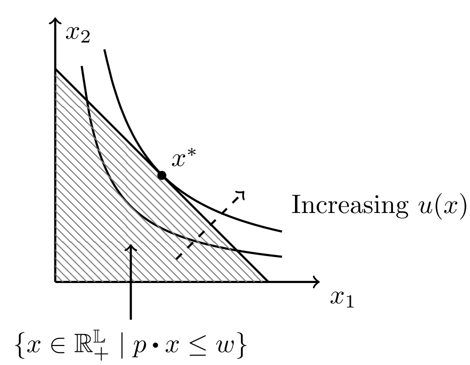
Utility Maximization Problem

Additionally assume $\succsim$ is locally nonsatiated, $p \gg 0$, and $w > 0$

Utility Maximization Problem: $\max\limits_{x \geq 0} \; u(x) \;\; s.t. \;\; p \bullet x \leq w$

Proposition 2.6: If $p \gg 0$ and $u(\cdot)$ is continuous, then the UMP has a solution.

- If $p \gg 0$, then $B_{p,w}$ is a compact set, and a continuous function always has a maximum value on it



Walrasian Demand Correspondence / Function

$\boldsymbol{x(p,w)}$ is the solution to the UMP (the x which achieves maximum utility)

Proposition 2.7: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$. Then the Walrasian demand correspondence $x(p,w)$ possesses the following properties:

(i) **Homogeneous of degree zero:** $x(\alpha p, \alpha w) = x(p,w)\ \ \forall p, w, \alpha > 0$

- The feasible set of consumption bundles is the same when prices and wealth are multiplied by α, so the set of maximizers must also be the same

(ii) **Walras' law:** $p \bullet x = w\ \ \forall x \in x(p,w)$

- Follows from local nonsatiation

(iii) **Convexity/uniqueness:** If $\succsim$ is convex ($u(\cdot)$ is quasiconcave), then $x(p,w)$ is a convex set. If $\succsim$ is strictly convex ($u(\cdot)$ is strictly quasiconcave), then $x(p,w)$ consists of a single element (is the Walrasian demand function).

- Follows from the intersection points of convex or strictly convex $\succsim$ with the budget line (intersection of two convex sets is convex)

By Kuhn-Tucker, $x^* \in x(p,w)$ is a solution to the UMP iff there exists a Lagrange multiplier $\lambda \geq 0$ such that for all $\ell = 1, ..., L$:

$$\frac{\partial u(x^*)}{\partial x_\ell} \leq \lambda p_\ell \text{ with equality if } x_\ell^* > 0$$

We can use this formula when $u(\cdot)$ is differentiable and quasiconcave (so the KT conditions are sufficient), $u(\cdot)$ is increasing (so the budget constraint is binding), and $\partial u^*(x)/\partial x_\ell \neq 0\ \ \forall \ell$ (the agent cares about all goods).

When $x \gg 0$:

$$\nabla u(x) = \begin{bmatrix} \partial u(x)/\partial x_1 \\ \vdots \\ \partial u(x)/\partial x_L \end{bmatrix} = \lambda \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix}$$

$$\frac{\partial u(x^*)/\partial x_k}{\partial u(x^*)/\partial x_\ell} = MRS_{k,\ell} = \frac{p_k}{p_\ell} \quad \forall k, \ell$$

Finding Walrasian Demand

If $\boldsymbol{u(\cdot)}$ is differentiable:

- ❊ Take $MRS_{12} = \dfrac{\partial u(x^*)/\partial x_1}{\partial u(x^*)/\partial x_2} = \dfrac{p_1}{p_2}$ (use a monotonic transformation e.g., ln to simplify)
- ❊ Solve for x_2 as a function of x_1
- ❊ Substitute x_2 into the budget constraint $w = p_1 x_1 + p_2 x_2$ and solve for x_1, and then x_2
- ❊ Check that $x_\ell \geq 0\ \ \forall \ell$!!! Otherwise, use piecewise demand functions (think about Kuhn-Tucker)
- ❊ If $\partial u(x^*)/\partial x_\ell$ has infinite marginal utility at 0 (x_ℓ alone in the denominator), then the agent always wants some of that good (can rule out a corner w.r.t. that good)

- ❊ If the agent has MRS that is never equal to the price ratio, then they will demand only the good with the higher marginal utility per dollar
- ❊ If the agent has MRS that is always equal to the price ratio, then they will demand any combination of the goods
- ❊ Watch out for indifference towards a good or wanting less of a good
- ❊ Cobb-Douglas: $\ln(u) = \alpha \ln x_1 + (1-\alpha)\ln x_2 \quad x_1^* = \dfrac{\alpha w}{p_1} \qquad x_2^* = \dfrac{(1-\alpha)w}{p_2}$

If $u(\cdot)$ is not differentiable:

- ❊ Draw a picture and find the highest indifference curve (e.g., min function)

Check that the demand functions are HOD 0

Indirect Utility Function

$\boldsymbol{v(p,w)}$ is the utility generated by solution to the UMP (the value of $u(x^*)$)

- When calculating, use the original utility function (NOT a monotonic transformation)

Proposition 2.8: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$. Then the indirect utility function $v(p,w)$ possesses the following properties:

(i) **Homogeneous of degree zero:** $v(\alpha p, \alpha w) = v(p,w) \;\; \forall p, w, \alpha > 0$

(ii) **Nonincreasing** in p_ℓ $\forall \ell$ and **increasing** in w

(iii) **Quasiconvex:** the set $\{(p,w) \mid v(p,w) \leq \bar{v}\}$ is convex for any $\bar{v}$

(iv) **Continuous** in p and w

Indirect Utility ⇒ Walrasian Demand

Roy's Identity: Suppose that $x(p,w)$ is a singleton, that v is differentiable at $(p,w) \gg 0$, and that $\frac{\partial v}{\partial w} > 0$.

$$x_\ell(p,w) = \frac{-\,\partial v(p,w)/\partial p_\ell}{\partial v(p,w)/\partial w} \qquad \forall \ell = 1, \ldots, L$$

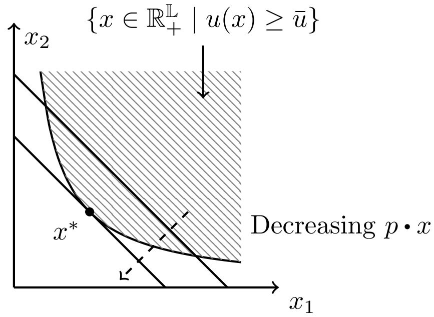
Expenditure Minimization Problem

Additionally assume $\succsim$ is locally nonsatiated, $p \gg 0$, and $\bar{u} > u(0)$ where $u(0)$ is the utility from consuming bundle $x = (0, 0, ..., 0)$

Expenditure Minimization Problem: $\min\limits_{x \geq 0} \;\; p \cdot x \;\; s.t. \;\; u(x) \geq \bar{u}$

Proposition 2.10: If $p \gg 0$ and $u(\cdot)$ is continuous and there is some x such that $u(x) \geq \bar{u}$, then the EMP has a solution

- $u(x) \geq \bar{u}$ is a non-empty closed set, and $p \bullet x$ is decreasing in x



Hicksian Demand Function / Correspondence

$\boldsymbol{h(p, \bar{u})}$ is the solution to the EMP (the x which achieves minimum expenditure)

Proposition 2.11: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$. Then the Hicksian demand correspondence $h(p, \bar{u})$ possesses the following properties:

(i) **Homogeneous of degree zero in *p*:** $h(\alpha p, \bar{u}) = h(p, \bar{u})\ \forall p, \bar{u}, \alpha > 0$

(ii) **No excess utility:** $u(x) = \bar{u}\ \forall x \in h(p, \bar{u})$

(iii) **Convexity/uniqueness:** If $\succsim$ is convex ($u(\cdot)$ is quasiconcave), then $h(p, \bar{u})$ is a convex set. If $\succsim$ is strictly convex ($u(\cdot)$ is strictly quasiconcave), then $h(p, \bar{u})$ consists of a single element (is the Hicksian demand function).

Expenditure Function

$\boldsymbol{e(p, \bar{u})}$ is the expenditure generated by solution to the EMP (the value of $p \bullet x^*$)

Proposition 2.12: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$. Then the expenditure function $e(p, \bar{u})$ possesses the following properties:

(i) **Homogeneous of degree one in *p*:** $e(\alpha p, \bar{u}) = \alpha e(p, \bar{u})\ \forall p, \bar{u}, \alpha > 0$

(ii) **Nondecreasing** in p_ℓ $\forall \ell$ and **increasing** in $\bar{u}$

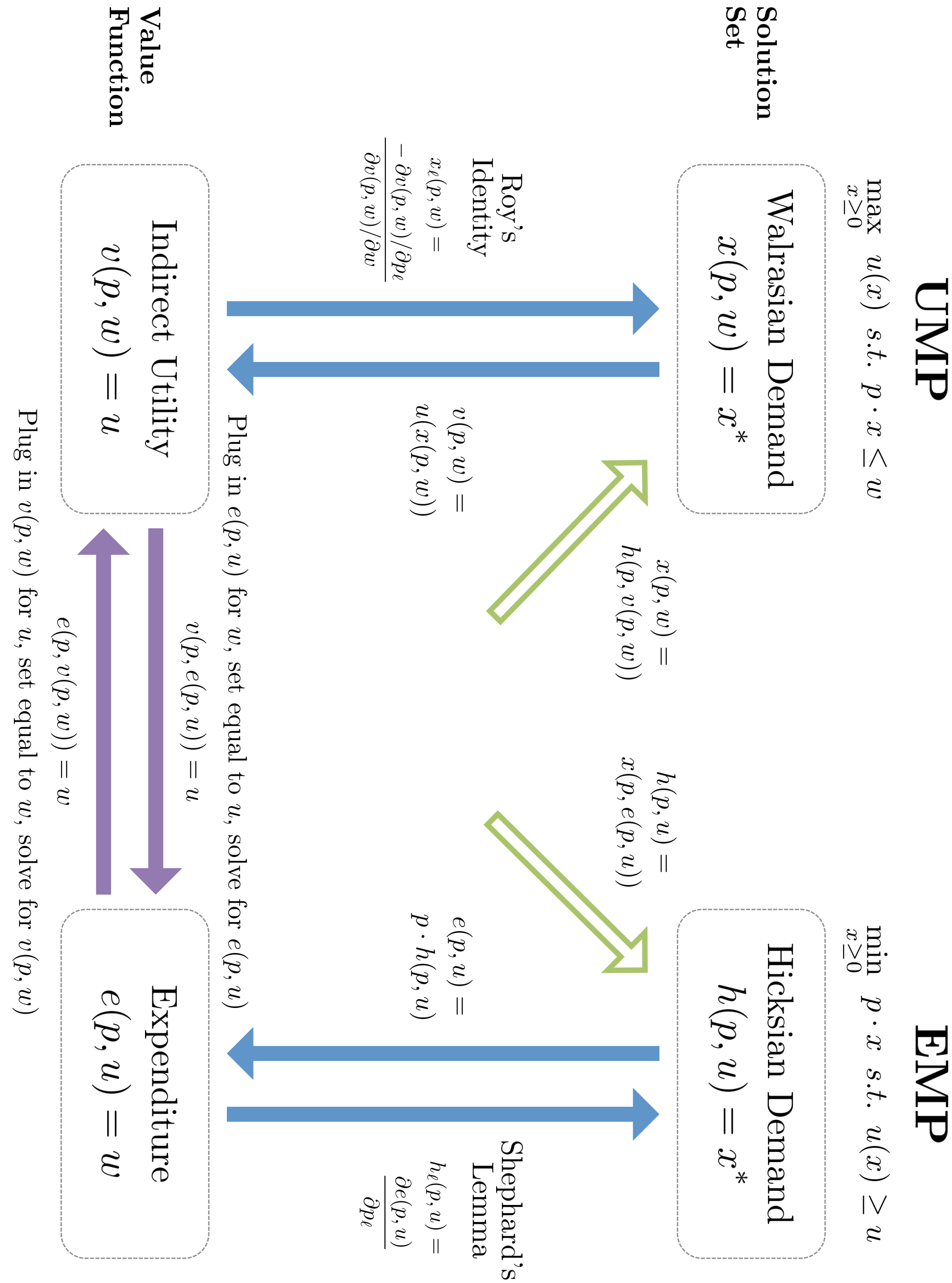
(iii) **Concave in *p*:** $e(\alpha p + (1-\alpha)p', \bar{u}) \geq \alpha e(p, \bar{u}) + (1-\alpha)e(p', \bar{u})$

(iv) **Continuous** in p and $\bar{u}$

Expenditure ⇒ Hicksian Demand

Shephard's Lemma: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ on $\mathbb{R}^L_+$ and suppose that $h(p, \bar{u})$ is is a singleton.

$$h_\ell(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_\ell} \quad \forall \ell = 1, \ldots, L$$



* If using piece-wise functions, flip around constraints at $e(p,u)$ (put in terms of u instead of w)

Hicksian Comparative Statics

A **Hicksian compensated price change** is accompanied by a wealth change so that the initial utility level is just achievable.

- A Slutsky compensated price change adjustes wealth so that the consumer can afford the previous bundle (which may give them an increase in utility if they choose a different bundle); a Hicksian compensated price adjustes wealth to put them on the same indifference curve
- We can do this kind of compensated price change now that we have a concept of "indifference curve"

Compensated Law of Demand (Hicksian): Suppose $p, p' \geq 0$ and let $x \in h(p,u)$ and $x' \in h(p',u)$. Then $[p' - p] \cdot [x' - x] \leq 0$

- Substitution effect is negative, and now there is no income effect
- In particular, $h_k(p,u)$ is non-increasing in p_k – if a good becomes more expensive, you want weakly less of it (assuming single-valued)

$$D_p h(p,u) = \begin{bmatrix} \frac{\partial h_1(p,u)}{\partial p_1} & \cdots & \frac{\partial h_1(p,u)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_L(p,u)}{\partial p_1} & \cdots & \frac{\partial h_L(p,u)}{\partial p_L} \end{bmatrix}$$

Proposition 2.15: Suppose that $u(\cdot)$ represents a preference relation $\succsim$ and that $h(p,u)$ is a singleton and continuously differentiable at (p,u) where $p \gg 0$. Then:

(i) $D_p h(p,u)$ is negative semi-definite

(ii) $D_p h(p,u)$ is symmetric

Slutsky Equation

Proposition 2.17: Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succsim$. Let $p \gg 0$ and $w = e(p,u)$. If $h(p,u)$ and $x(p,w)$ are single-valued and differentiable at (p,u,w), then for all ℓ, k:

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} \, x_k(p,w)$$

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) \; x(p,w)^T$$

- Substitution Effect = Total Effect + Income Effect
- Now that we have the concept of indifference curves, we can isolate the substitution effect directly with $D_p h(p,u)$ rather than the Slutsky matrix
- With our additional restrictions, the matrix is now symmetric

Consumer Welfare: Price Changes

We use the expenditure function to measure welfare changes in dollars, when prices change from p to p'

We assume wealth is fixed, so $e(p,u) = e(p',u') = w$

Compensating Variation: $CV = e(p,u) - e(p',u) = w - e(p',u)$

- The change in wealth required to achieve original utility
- > 0 means the price change improved welfare
- If only p_ℓ changes, $CV = \int_{p'_\ell}^{p_\ell} h_\ell(p,u)\ dp_\ell$

Equivalent Variation: $EV = e(p,u') - e(p',u') = e(p,u') - w$

- The change in wealth that would be equivalent to the price change
- > 0 means the price change improved welfare
- If only p_ℓ changes, $EV = \int_{p'_\ell}^{p_\ell} h_\ell(p,u')\ dp_\ell$

Consumer Surplus: $CS = \int_{p'_\ell}^{p_\ell} x_\ell(p,w)\ dp_\ell$

- Normal goods: $CV \leq CS \leq EV$
- Inferior goods: $EV \leq CS \leq CV$

Consumer Welfare: Price Indices

Ideal Index $= \dfrac{e(p',u)}{e(p,u)}$

Laspeyres Index $= \dfrac{p' \cdot x}{p \cdot x} \geq \dfrac{e(p',u)}{e(p,u)} = Ideal(u)$

Paasche Index $= \dfrac{p' \cdot x'}{p \cdot x'} \leq \dfrac{e(p',u')}{e(p,u')} = Ideal(u')$

Laspeyres and Paasche both suffer from substitution bias, because they do not account for the fact that when prices change, consumers will substitute to cheaper products.

3. General Equilibrium

(Reference: Chapters 15, 16, 17, Microeconomic Theory, Mas-Colell, Whinston, & Green)

The Walrasian Model

- L commodities with market prices $p \in \mathbb{R}^L_+$
- Pure-exchange economy $\mathcal{E} = (X^i, u^i, e^i)_{i=1}^n$ giving for each agent $i \in \{1, \dots, n\}$:
  - **Consumption set** X^i (we assume $X^i = \mathbb{R}^L_+ \ \forall i$)
  - **Utility function** $u^i : \mathbb{R}^L_+ \to \mathbb{R}$
  - **Endowment bundle** $e^i \in \mathbb{R}^L_+$
- Given prices, an agent has budget set $\mathcal{B}^i(p) = \{x \mid p \bullet x \leq p \bullet e^i\}$

An **allocation** $x = (x^1, x^2, \dots, x^n)$ denotes a $L \times 1$ consumption bundle for each agent

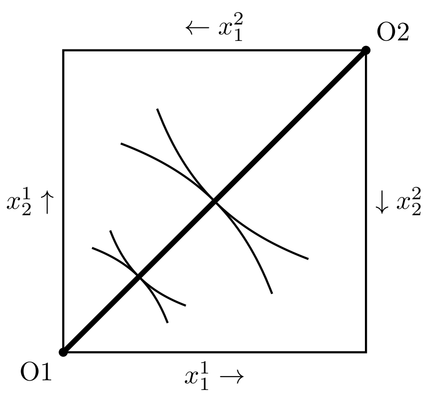
An allocation is **feasible** if for all $\ell \in 1, \dots, L$ $\quad \sum_{i=1}^n x^i_\ell \leq \sum_{i=1}^n e^i_\ell$

For the $L = 2, n = 2$ case we represent the set of feasible allocations with the Edgeworth Box.

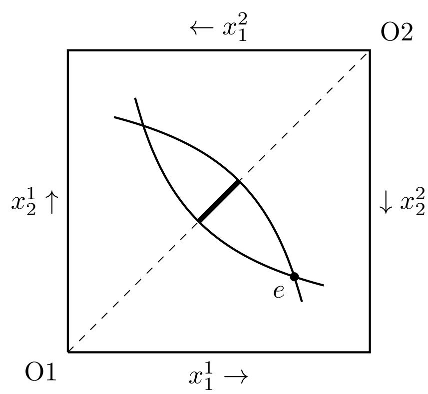
Pareto Optimality

A feasible allocation x^* is **pareto optimal** if there is no other feasible allocation $\hat{x}$ such that $u^i(\hat{x}^i) \geq u^i(x^{i*})$ for all $i = 1, \dots, n$, with $u^i(\hat{x}^i) > u^i(x^{i*})$ for some i

- The pareto optimal set is unrelated to endowments
- The **contract curve** is a subset of the pareto optimal points defined by e, where both agents have the same or better utility as that given to them by e (we expect agents to trade to some point on the contract curve, though we don't know to where because bargaining power may not be equal)



Pareto optimal set
(Cobb-Douglas utility)



Contract curve for some e

Weighted Welfare Maximum: Suppose $u^i(\cdot)$ is increasing, continuous, and concave $\forall i$. An allocation $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ is pareto optimal $\Leftrightarrow$ there exists **pareto weights** $\lambda \in \mathbb{R}^n_+, \lambda \neq 0$ such that $\bar{x}$ solves:

$$\max_x \sum_{i=1}^n \lambda^i u^i(x) \quad s.t. \quad \sum_{i=1}^n x^i = \sum_{i=1}^n e^i \quad x^i \in X^i \ \forall i$$

- If utility is quasilinear in the same (private) good across all agents, then the parteo weights must all be equal

Finding Set of Pareto Optimal Allocations

Interior PO allocations:

- Find allocations where indifference curves are tangent
- If $u^i(\cdot)$ is increasing and differentiable $\forall i$, then the interior PO set is characterized by:
  - $\frac{MU_h^1(x^1)}{MU_k^1(x^1)} = \frac{MU_h^i(x^i)}{MU_k^i(x^i)} \quad \forall i \in 2, \dots, n \quad \forall h, k \in 1, \dots, L, \ \ h \neq k$
  - $\sum_{i=1}^n x_\ell^i = \sum_{i=1}^n e_\ell^i \quad \forall \ell \in 1, \dots, L$
- For $L = 2, n = 2$: $MRS^1 = MRS^2$

Corner PO allocations:

- Allocations where one agent has everything (the two origins in the Edgeworth Box) are almost always PO
- The PO set is almost always a continuous set: check the edges which connect the origins and the interior PO set
- Edge PO allocations arise when the agents would prefer to adjust to where their indifference curves are tangent, but cannot due to non-negativity constraints

Walrasian Equilibrium

A **Walrasian equilibrium** is prices p^* and allocation $x^* = (x^{i*})_{i=1}^n$ such that:

(i) Every agent is maximizing their utility given p^*: $x^{i*} \in \arg\max_{x \in \mathcal{B}^i(p)} u^i(x) \quad \forall i$

(ii) Markets clear: $\sum_{i=1}^n x_\ell^i = \sum_{i=1}^n e_\ell^i \quad \forall \ell \in 1, \dots, L$

Notes:

- A Walrasian equilibrium takes endowments as given
- In a Walrasian equilibrium, there is no excess supply or excess demand (the optimal bundles must be at the same point in the Edgeworth box)

- Multiple Walrasian equilibria are possible; there also may be no Walrasian equilibria
- In a competitive equilibrium, price-taking is automatic; in a Walrasian equilibrium, price-taking is not automatic (e.g., in the case of finitely many consumers) but assumed

Finding Walrasian Equilibria:

1. Solve individual Walrasian demand functions
   a. If needed, use a piecewise function based on price ratios to ensure that $x^* \geq 0$
2. Set aggregate demand for one good equal to aggregate endowment for that good ($x_1^1 + x_1^2 = e_1^1 + e_1^2$)
3. Replace w with cash value of endowments
4. Assert the price of one good = 1 (numeraire good)
5. Solve for the other price
   a. Quadratic Formula: $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
6. Use prices and demand functions to check that markets for other goods also clear

At an interior equilibrium: $MRS_{12}^1 = \dfrac{\partial u^1/\partial x_1^1}{\partial u^1/\partial x_2^1} = \dfrac{\partial u^2/\partial x_1^2}{\partial u^1/\partial x_2^2} = MRS_{12}^2 = \dfrac{p_1}{p_2}$

Look Out For:

- An agent prefers less of a good
- An agent doesn't care about a good
- Goods are perfect complements for an agent
- An agent has increasing marginal utility (will want all of that good)
- Corner solutions
  - Check all four sides & corners of the Edgeworth box for corner solutions
  - We can rule out a side if an agent has infinite marginal utility in a good at 1 (the agent always wants a little bit of that good)
  - We can also rule out a side if that good = 0 implies a price (e.g., with piecewise demand functions) which means the agents have different optimal bundles (e.g., both agents want all of a good)
  - There may be a corner solution for which a range of prices will work (the price vector can take a range of slopes and still intersect the corner solution and be between two indifference curves, since the edge of the Edgeworth box cuts off the point of exact tangency)

Welfare Theorems

Key Assumptions: For all agents $i \in \mathcal{I}$:

(A1) $u^i(\cdot)$ is continuous

(A2) $u^i(\cdot)$ is increasing, i.e., $u^i(x') > u^i(x)$ whenever $x' \gg x$

(A3) $u^i(\cdot)$ is concave

(A4) $e^i \gg 0$

First Welfare Theorem

Let $(p, (x^i)_{i\in\mathcal{I}})$ be a Walrasian equilibrium for economy $\mathcal{E}$. If (A2) holds, the allocation $(x^i)_{i\in\mathcal{I}}$ is pareto optimal.

Second Welfare Theorem

Let $\mathcal{E}$ be an economy that satisfies (A1) – (A4). If $(x^i)_{i\in\mathcal{I}}$ is pareto optimal, then there exists a reallocation of endowments e' and a price vector p such that $(p, (x^i)_{i\in\mathcal{I}})$ is a Walrasian equilibrium of $\mathcal{E}'$.

- If you desire a particular pareto optimal outcome, find the price vector that goes through it and redistribute goods to some e' on that line or wealth so that the outcome is affordable
- To find needed wealth transfers (or transfers in units of the numeraire good), find p. Then
  $T^i = p \bullet x^i - p \bullet e^i$ (total transfers must sum to 0)

4. Monotone Comparative Statics

(Reference: pp. 33-40, Useful Math for Microeconomics, Jonathan Levin & Antonio Rangel)

Monotone comparative statics are a powerful tool for understanding whether the solution to a constrained optimization problem is nondecreasing or nonincreasing, without making assumptions about continuity, differentiability, or concavity of the function.

$x^*(\theta) = \arg\max_{x \in \mathcal{D}} f(x, \theta)$ for $\theta \in \Theta$

- x is the choice variable, θ is a parameter
- $f : \mathbb{R}^2 \to \mathbb{R}$
- Assume $x^*(\theta)$ is non-empty for all $\theta \in \Theta$ (may be multi-valued)

$A \leq_s B$ in the **strong set order:** $\quad a \in A \ \wedge \ b \in B \ \Rightarrow \ \min\{a, b\} \in A \ \wedge \ \max\{a, b\} \in B$

$x^*(\cdot)$ is **nondecreasing (in the strong set order)** in θ: $\quad \theta < \theta' \ \Rightarrow \ x^*(\theta) \leq_s x^*(\theta')$

- If $x^*(\cdot)$ is a compact-valued correspondence and nondecreasing in the strong set order, then $\bar{x}^*(\theta) = \max_{x \in x^*(\theta)} x$ and $\underline{x}^*(\theta) = \min_{x \in x^*(\theta)} x$ are also nondecreasing

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is **supermodular** in (x, θ) if for all $x' > x$ and $\theta' > \theta$:

- $f(x', \theta) - f(x, \theta)$ is nondecreasing in θ: $\quad f(x', \theta') - f(x, \theta') \geq f(x', \theta) - f(x, \theta)$
- $f(x, \theta') - f(x, \theta)$ is nondecreasing in x: $\quad f(x', \theta') - f(x', \theta) \geq f(x, \theta') - f(x, \theta)$ (equivalently)
- "Increasing differences"
- **Strictly supermodular** uses increasing and $>$

Proposition 4.1: A twice continuously differentiable $f(\cdot)$ is supermodular in (x, θ) $\ \Leftrightarrow \ f_{x\theta}(x, \theta) \geq 0 \ \forall (x, \theta)$

- $f_{x\theta} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial \theta}\right) = \dfrac{\partial}{\partial \theta}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x \partial \theta}$
- f is strictly supermodular $\ \Leftrightarrow \ f_{x\theta}(x, \theta) > 0 \ \forall (x, \theta)$

Topkis' Monotonicity Theorem: If $f(\cdot)$ is supermodular in (x, θ), then $x^*(\theta)$ is nondecreasing in θ

- To show nonincreasing in θ, then we need supermodular in $(x, -\theta)$

Proposition 4.4: If $f(\cdot)$ is strictly supermodular in (x, θ), then for all $\theta' > \theta, x \in x^*(\theta), x' \in x^*(\theta')$, we have that $x' \geq x$

- If $f(\cdot)$ is continuously differentiable in x and x is in the interior of the fixed choice set $\mathcal{D}$, then $x' > x$

Single-Crossing

Supermodularity is a stronger assumption than what is required for $x^*(\theta)$ to be nondecreasing.

The function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is **single-crossing** in (x, θ) if for all $x' > x$ and $\theta' > \theta$:

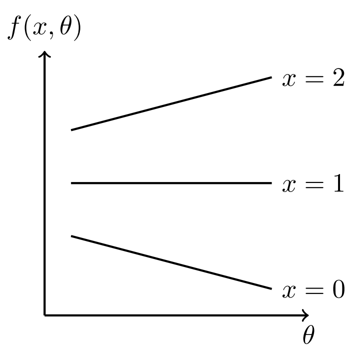
(i) $f(x', \theta) - f(x, \theta) \geq 0 \;\Rightarrow\; f(x', \theta') - f(x, \theta') \geq 0$

(ii) $f(x', \theta) - f(x, \theta) > 0 \;\Rightarrow\; f(x', \theta') - f(x, \theta') > 0$

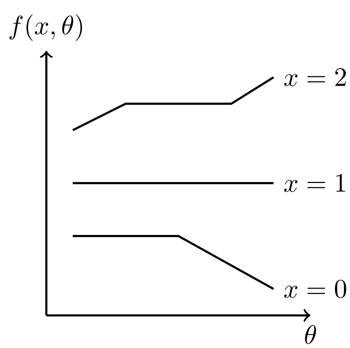
$f(\cdot)$ is supermodular $\;\Rightarrow\;$ $f(\cdot)$ is single-crossing

$f(\cdot)$ is single-crossing $\;\Rightarrow\;$ $x^*(\theta)$ is nondecreasing

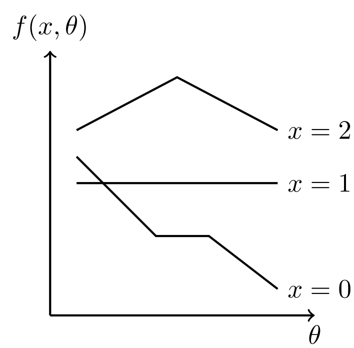
$x^*(\theta)$ is nondecreasing in θ for all choice sets $\mathcal{D}$ $\;\Rightarrow\;$ $f(\cdot)$ is single-crossing



Strictly supermodular
(Discrete x)



Supermodular



Single-crossing

Parameterization

Compare solutions to two distinct maximization problems, $\max g(x)$ and $\max h(x)$. Let $\theta \in \{0, 1\}$ and:

$$f(x, \theta) = \begin{cases} g(x) & \text{if } \theta = 0 \\ h(x) & \text{if } \theta = 1 \end{cases}$$

If $f(\cdot)$ is supermodular (i.e., if $h(x) - g(x)$ is nondecreasing in x), then $x^*(1) \geq x^*(0)$

Aggregation

$$\max_{x\in\mathbb{R}, z\in\mathbb{R}^k, x\leq F(z)} px - w \bullet z$$

If we only care about how p impacts $x^*(p, w)$, we rewrite into:

- Maximization problem given fixed x that does not involve p: $-c(x) = \max_{z\in\mathbb{R}^k, x\leq F(z)} -w \bullet z$
- Maximization problem with x as the only choice variable: $\max_{x\in\mathbb{R}} px - c(x)$

Now it is easy to show that the latter maximization problem is supermodular in (x, p):

- $[px' - c(x')] - [px - c(x)] \;=\; p(x' - x) - [c(x') - c(x)]$
- The first term is nondecreasing in p, and the second is unrelated to p
- So the objective function is supermodular (x, p) and thus $x^*(p)$ is nondecreasing in p

Implicit Function Theorem

With additional assumptions we can use the Implicit Function Theorem (more restrictive).

$\max_{x\in\mathbb{R}} f(x, \theta)$ where $\theta \in \mathbb{R}$

- $f(\cdot)$ is twice continuously differentiable and strictly concave: $f_{xx} < 0$
- There is a unique solution for every θ characterized by the FOC: $f_x(x^*(\theta), \theta) = 0$
- Totally differentiate with respect to θ: $\dfrac{\partial f_x(x^*(\theta), \theta)}{\partial x} \cdot \dfrac{\partial x^*(\theta)}{\partial \theta} + \dfrac{\partial f_x(x^*(\theta), \theta)}{\partial \theta} = 0$
- Simplify: $\dfrac{\partial x^*(\theta)}{\partial \theta} = \dfrac{f_{x\theta}(x^*(\theta), \theta)}{-f_{xx}(x^*(\theta), \theta)}$
- $x^*(\theta)$ is nondecreasing $\;\Leftrightarrow\;$ $f_{x\theta}(x^*(\theta), \theta) \geq 0$

5. Production

(Reference: Chapter 5, Microeconomic Theory, Mas-Colell, Whinston, & Green)

Consider an economy with L commodies.

Production plan: $y = (y_1, ..., y_L) \in \mathbb{R}^L$, where $y_k > 0$ is an output and $y_k < 0$ is an input

Production set: $Y \subset \mathbb{R}^L$ is the set of all production plans that are feasible for the firm based on technology

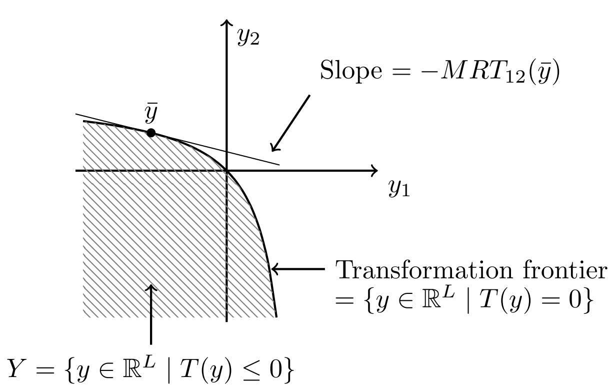
Y can be described using **transformation function** $T : \mathbb{R}^L \to \mathbb{R}$, where
$Y = \{y \in \mathbb{R}^L \mid T(y) \leq 0\}$, and $T(y) = 0 \;\; \Leftrightarrow \;\; y$ is an element of the boundary of Y

Marginal rate of transformation: If $T(\cdot)$ is differentiable and $T(y) = 0$ for some production plan y, then for any commodities k and ℓ: $MRT_{k\ell}(y) = \dfrac{\partial T(y)/\partial y_k}{\partial T(y)/\partial y_\ell}$

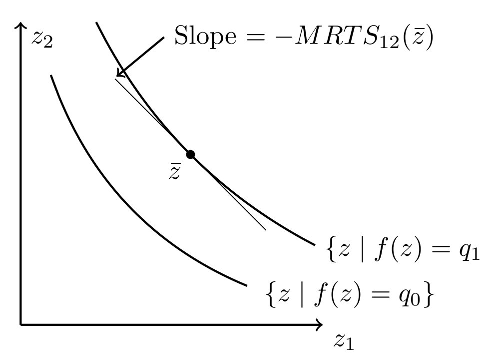
When goods can be either inputs or outputs but not both, we denote outputs by $q = (q_1, ..., q_H)$ and inputs by $z = (z_1, ..., z_K)$ where $H + K = L$ and $q_i \geq 0, z_i \geq 0 \;\; \forall i$

Single-output case: $q = f(z)$ where $f(z)$ is the production function (isoquants $\approx$ indifference curves)

Marginal rate of technological substitution: $MRTS_{k\ell}(z) = \dfrac{\partial f(z)/\partial z_k}{\partial f(z)/\partial z_\ell}$
(Special case of $MRT_{k\ell}$ for single-output case)



General case



Single-output case

Common Potential Properties of Production Set Y

* **Nonempty:** The firm can do something

* **Closed:** The set Y includes its boundary

* **Free disposal:** $y \in Y \ \wedge \ y' \leq y \ \Rightarrow \ y' \in Y$ – extra inputs or outputs can be disposed of at no cost

No free lunch: $y \in Y \ \wedge \ y \geq 0 \ \Rightarrow \ y = 0$ – not possible to produce something from nothing

Shut down: $0 \in Y$ – complete shutdown is possible (opposite of sunk costs)

Nonincreasing returns to scale: $y \in Y \ \Rightarrow \ \alpha y \in Y \ \forall \alpha \in [0,1]$ – any feasible production plan can be scaled down (implies shut down)

Nondecreasing returns to scale: $y \in Y \ \Rightarrow \ \alpha y \in Y \ \forall \alpha \geq 1$ – any feasible production plan can be scaled up (consistent with sunk costs or fixed setup costs)

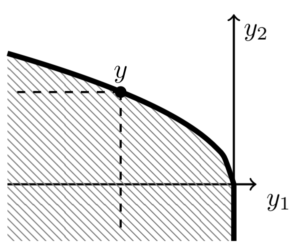
Constant returns to scale: $y \in Y \ \Rightarrow \ \alpha y \in Y \ \forall \alpha \geq 0$ – any feasible production plan can be scaled down or scaled up ($\Leftrightarrow$ nonincreasing RTS $\wedge$ nondecreasing RTS)

- In the single output case, Y satisfies constant returns to scale $\Leftrightarrow$ $f(z)$ is HOD 1

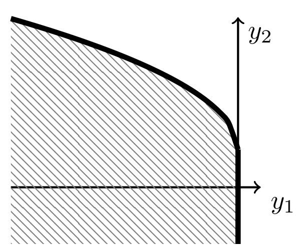
Y is a convex set: $y, y' \in Y \ \wedge \ \alpha \in [0,1] \ \Rightarrow \ \alpha y + (1-\alpha)y' \in Y$

- "Unbalanced" input combinations are not more productive than "balanced" ones
- Shut down $\wedge$ convexity $\Leftrightarrow$ nonincreasing returns to scale
- In the single output case, Y is a convex set $\Leftrightarrow$ $f(z)$ is concave

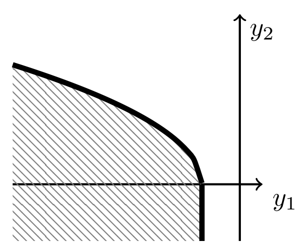
For the special case of one-input, one-output:



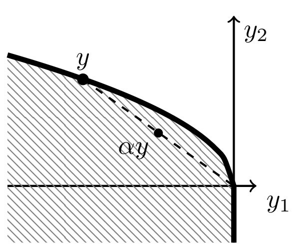
Free disposal



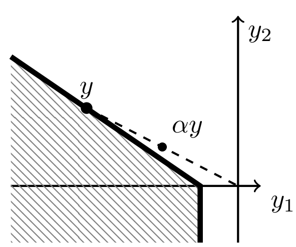
Violates no free lunch



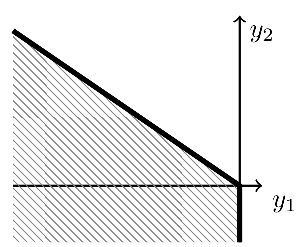
Sunk costs (violates shut down)



Nonincreasing RTS



Nondecreasing RTS



Constant RTS

Profit Maximization Problem

Assume that $p \gg 0$, prices are taken as given, and that Y satisfies nonemptiness, closedness, and free disposal.

Profit Maximization Problem: $\max_y \; p \bullet y \;\; s.t. \; T(y) \leq 0$

- = revenue − cost = $p \bullet y(+)$ - $p \bullet y(-)$
- We assume that the necessary conditions on Y are in place so that a maximum exists
- For some $\lambda \geq 0$, $p_\ell = \lambda \dfrac{\partial T(y^*)}{\partial y_\ell} \; \forall \ell$
- For interior solution, $MRT_{12}(y^*) = \dfrac{\partial T(y^*)/\partial y_1}{\partial T(y^*)/\partial y_2} = \dfrac{p_1}{p_2}$

Optimal Production Correspondence (Supply Correspondence)

$\boldsymbol{y(p)}$ is the solution to the PMP (the y which achieves maximum profit)

- $y(p) = \{y \in Y \mid p \bullet y = \pi(p)\}$
- $y(p)$ may be single-valued or multi-valued

Proposition 5.2: The optimal production correspondence y has the following properties:

(i) y is homogeneous of degree 0 − $y(\alpha p) = y(p) \;\; \alpha > 0$

(ii) If Y is a convex set, then for all p, the set $y(p)$ is convex. If Y is a strictly convex set, $p \neq 0$ and $y(p) \neq \emptyset$, then $y(p)$ is a singleton.

(iii) The Law of Supply: For any $p, p', y \in y(p), y' \in y(p')$: $(p' - p)(y' - y) \geq 0$

Profit Function

$\boldsymbol{\pi(p)}$ is the profit generated by the solution to the PMP (the value of $p \bullet y^*$)

Proposition 5.1: The profit function π has the following properties:

(i) **Homogeneous of degree one:** $\pi(\alpha p) = \alpha \pi(p) \;\; \forall \alpha > 0$

(ii) **Convex:** $\alpha \pi(p) + (1 - \alpha)\pi(p') \geq \pi(\alpha p + (1 - \alpha)p')$

(iii) If Y is closed and a convex set, then $Y = \{y \in \mathbb{R}^L \mid p \bullet y \leq \pi(p) \;\; \forall p \in \mathbb{R}^L\}$

(iv) If Y is closed and a convex set and has the free disposal property, then $Y = \{y \in \mathbb{R}^L \mid p \bullet y \leq \pi(p) \;\; \forall p \in \mathbb{R}^L_+\}$

Single Output Case

Profit Maximization Problem: $\max_{z \geq 0} \; pf(z) - w \bullet z$

- Inputs z, input prices $w \gg 0$, output quantity $f(z)$, at price p
- $\forall \ell \quad p\, \dfrac{\partial f(z^*)}{\partial z_\ell} \leq w_\ell$ with equality if $z_\ell^* > 0$
- For all inputs actually used, $\dfrac{\partial f(z^*)}{\partial z_\ell} = \dfrac{w_\ell}{p}$ and $MRTS_{\ell,k} = \dfrac{w_\ell}{w_k}$

Optimal Production & Profit Function

Proposition 5.3: Assume that Y is closed and satisfies free disposal.

(i) **Hotelling's Lemma (Traditional):** If y is single-valued in the neighborhood of p, then π is differentiable at p and $\dfrac{\partial \pi(p)}{\partial p_\ell} = y_\ell(p) \;\; \forall \ell$

(ii) **Hotelling's Lemma (Producer Surplus):** If y is non-empty valued and $\hat{y}(p) \in y(p) \;\; \forall p$, then for all p', p'': $\quad \pi(p_j', p_{-j}') - \pi(p_j'', p_{-j}') = \displaystyle\int_{p_j''}^{p_j'} \hat{y}(s, p_{-j}')ds$

(iii) If y is single-valued and continuously differentiable, the matrix $D_p y(p) = D_p^2 \pi(p)$ is symmetric and positive semidefinite, with $[D_p y(p)]p = 0$

Cost Minimization Problem

Focus on the single-output case. Assume z is non-negative vector of inputs, q is the amount of output, $w \gg 0$ the vector of input prices, and there is free disposal of output.

Cost Minimization Problem: $\min_{z \in \mathbb{R}_+^K} \; w \cdot z \;\; s.t. \;\; f(z) \geq q$

- Minimizes cost given quantity q
- Analogous to the EMP
- For some $\lambda \geq 0, \lambda \dfrac{\partial f(z)}{\partial z_k} \leq w_k$ with equality if $z_k > 0$
- $\lambda = \dfrac{\partial c(w,q)}{\partial q} =$ the marginal cost of production
- With $f(\cdot)$ concave, profit maximization is a special case of cost minimization in which the shadow price of output is the market price p. When the production set has nondecreasing returns to scale ($f(\cdot)$ is not concave), the CMP is better behaved than the PMP (PMP only has 0 and ∞ as solutions)

Conditional Factor Demand Correspondence

$\boldsymbol{z(w,q)}$ is the solution to the CMP (the z which achieves minimum cost conditional on producing output q)

Cost Function

$\boldsymbol{c(w,q)}$ is the cost generated by the solution to the CMP (the value of $w \cdot z^*$)

Proposition 5.4: The cost function c has the following properties:

(i) c is homogeneous of degree 1 in w and is increasing in q

(ii) c is a concave function of w

(iii) If $f(\cdot)$ is concave, then c is a convex function of q (i.e., marginal costs are increasing in q)

(iv) Shephard's Lemma: If z is single-valued, then c is differentiable with respect to w and $\dfrac{\partial c(w,q)}{\partial w_k} = z_k(w,q) \;\; \forall k$

(v) If z is a differentiable function, then the matrix $D_w z(w,q) = D_w^2 c(w,q)$ is symmetric and negative semidefinite, and $[D_w z(w,q)]w = 0$

6. Choice Under Risk & Uncertainty

(Reference: Chapter 6, Microeconomic Theory, Mas-Colell, Whinston, & Green)

We consider choice under objectively known probabilities.

Expected Utility Theory

Let C be a finite set of N potential outcomes.

Simple lottery: A vector $L = (p_1, ..., p_N)$ with $p_n \geq 0\ \forall n$ and $\sum_n p_n = 1$ where p_n is the probability of outcome n occurring.

- A simple lottery can be represented as a point in the $N-1$ dimensional simplex,
  $\Delta = \{p \in \mathbb{R}^N_+ \mid p_1 + ... + p_N = 1\}$
- When $N = 3$, Δ is the space in 3D such that $p_1 + p_2 + p_3 = 1$ (an equilateral triangle connecting the points $(1,0,0), (0,1,0)$, and $(0,0,1)$)

Compound lottery: Given K simple lotteries, $L_k = (p_1^k, ..., p_N^k), k = 1, .., K$ and probabilities $\alpha_k \geq 0$ with $\sum_k \alpha_k = 1$, the compound lottery $(L_1, ..., L_k; \alpha_1, ..., \alpha_k)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1, ..., K$

- The outcome of the lottery is another lottery
- We can reduce any compound lottery to a simple lottery through vector addition
  - $L = \alpha_1 L_1 + ... + \alpha_K L_K$ – this is the weighted average and thus $L \in \Delta$ ($\mathcal{L}$ is a convex set)
  - The probability of outcome n is $p_n = \alpha_1 p_n^1 + ... + \alpha_K p_n^K$
- We assume that when faced with a complex lottery, only the reduced lottery over outcomes is relevant to the agent

<u>Preferences over Lotteries</u>

Let $\mathcal{L}$ be the (infinite) set of all simple lotteries over the (finite) set of outcomes C. We assume that the agent has a preference relation $\succsim$ on $\mathcal{L}$.

Continuity axiom: $\succsim$ on $\mathcal{L}$ is continuous if for all $L, L', L'' \in \mathcal{L}$, the sets $\{\alpha \in [0,1] \mid \alpha L + (1-\alpha)L' \succsim L''\}$ and $\{\alpha \in [0,1] \mid L'' \succsim \alpha L + (1-\alpha)L'\}$ are closed

- If the set is non-empty, then there exists an α such that you are indifferent (equivalent to previous definition of continuous $\succsim$ where upper and lower contour sets are closed)
- If $\succsim$ is complete and transitive, then the continuity axiom implies the existence of a utility function representing $\succsim$, i.e., $U : \mathcal{L} \to \mathbb{R}$ such that $L \succsim L' \;\Leftrightarrow\; U(L) \geq U(L')$
- $\Rightarrow\ \exists \alpha_L \ni L \sim \alpha_L \bar{L} + (1-\alpha_L)\underline{L}\ \forall L\ \ \alpha_L \in [0,1]$ where $\bar{L}$ is the best outcome and $\underline{L}$ is the worst

Independence axiom: $\succsim$ on $\mathcal{L}$ satisfies the independence axiom if for all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0,1)$ we have $L \succsim L' \;\Leftrightarrow\; \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$

- If we mix each of two lotteries with a third one, the preference ordering of the two mixtures is independent of the third lottery used
- The independence axiom doesn't work with bundles, because having apples might change how you feel about bananas (e.g., liking or hating a combination). But with uncertainty, you are ultimately just consuming one outcome or the other.
- Requires that indifference curves are straight, parallel lines

A utility function $U : \mathcal{L} \to \mathbb{R}$ has an **expected utility form** if there is an assignment of numbers $(u_1, ..., u_N)$ to the N outcomes such that for every simple lottery $L = (p_1, ..., p_N) \in \mathcal{L}$ we have $U(L) = p_1 u_1 + ... + p_N u_N$ (u_1 is the utility of getting outcome 1 for sure)

- Often called a von Neumann-Morgenstern expected utility function

Proposition 6.1: A utility function $U : \mathcal{L} \to \mathbb{R}$ has an expected utility form $\;\Leftrightarrow\;$ U is linear, i.e.,

$$U(\sum_{k=1}^{K} \alpha_k L_k) = \sum_{k=1}^{K} \alpha_k U(L_k)$$

for any K lotteries $L_k \in \mathcal{L}, k = 1, ..., K$, and probabilities $(\alpha_1, ..., \alpha_K) \geq 0, \sum_k \alpha_k = 1$

Expected Utility Theorem: Suppose that $\succsim$ on $\mathcal{L}$ is complete and transitive and satisfies the continuity axiom. $\succsim$ satisfies the independence axiom $\;\Leftrightarrow\;$ $\succsim$ is represented by a utility function that has the expected utility form

- $L \succsim L' \;\Leftrightarrow\; \displaystyle\sum_{n=1}^{N} u_n p_n \geq \sum_{n=1}^{N} u_n p'_n$
- The independence axiom implies $\exists! \;\; \alpha_L \ni L \sim \alpha_L \bar{L} + (1-\alpha_L)\underline{L} \;\; \forall L \;\; \alpha_L \in [0,1]$
- Thus $\succsim$ is represented by $U(L) = \alpha_L$, which satisfies $U(\beta L + (1-\beta)L') = \beta U(L) + (1-\beta)U(L')$ $\forall L, L' \in \mathcal{L}, \beta \in [0,1]$ so $U(\cdot)$ is linear

Proposition 6.2: Suppose that $U : \mathcal{L} \to \mathbb{R}$ represents $\succsim$ and has the expected utility form. $\widetilde{U} : \mathcal{L} \to \mathbb{R}$ also represents $\succsim$ and has the expected utility form $\;\Leftrightarrow\;$ there are scalars $\beta > 0$ and γ such that $\widetilde{U}(L) = \beta U(L) + \gamma \;\; \forall L \in \mathcal{L}$

- Expected utility is a cardinal property, and only increasing linear transformations preserve it

Money Lotteries & Risk Aversion

We treat money as a continuous variable, extending the expected utility theory to an infinite domain.

We can describe a lottery over money with the cumulative distribution function $F : \mathbb{R} \to [0, 1]$

- $F(x) = P(X \leq x) = \sum_{k=-\infty}^{x} P(X = k) \quad \text{or} \quad \int_{-\infty}^{x} f(t)\ dt$
- We take $\mathcal{L}$ to be the set of all $F(\cdot)$ over the interval $[0, \infty)$
- For the compound lottery $(L_1, ..., L_K; a_1, ..., a_K)$, $F(x) = \sum_k \alpha_k F_k(x)$

The **expected utility function of a lottery over money** is $U(F) = \mathbb{E}[u(x)] = \int u(x)\ dF(x)$

- $= \sum_x u(x)\ P(X = x) \qquad$ if $F(\cdot)$ is discrete
- $= \int u(x) f(x)\ dx \qquad$ if $F(\cdot)$ is continuous
- As before, $U(\cdot)$ is linear in $F(\cdot)$
- The **Bernoulli utility function** $u(x)$ extends $(u_1, ..., u_N)$ to the infinite domain and gives the utility of receiving x dollars for sure
- We assume that $u(\cdot)$ is continuous and increasing: $\ x' > x \ \Leftrightarrow \ u(x') > u(x)$

An agent is **risk averse** if for any lottery $F(\cdot)$, the expected payoff $\int x\ dF(x)$ with certainty is as least as good as the lottery $F(\cdot)$ itself

- $U(F) = \int u(x)\ dF(x) \ \leq \ u\left(\int x\ dF(x)\right) \ \ \forall F(\cdot)$
- By Jensen's Inequality, risk aversion $\Leftrightarrow$ $u(\cdot)$ is a concave function
- **Risk loving** $\geq$ $\Leftrightarrow$ $u(\cdot)$ is a convex function
- **Risk neutral** $=$ $\Leftrightarrow$ $u(\cdot)$ is a linear function (concave and convex)
- **Strictly risk neutral** $<$ $\Leftrightarrow$ $u(\cdot)$ is a strictly concave function

The **certainty equivalent** of $F(\cdot)$, denoted $c(F, u)$, is the amount of money for which the agent is indifferent between gamble $F(\cdot)$ and the certain amount

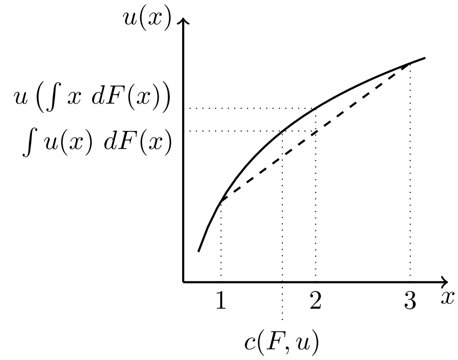
$$U(F) = \int u(x)\ dF(x) = u(c(F, u))$$

Proposition 6.4: Suppose a decision maker is an expected utility maximizer with a Bernoulli utility function $u(\cdot)$ on amounts of money. The following properties are equivalent:

(i) The decision maker is risk averse

(ii) $u(\cdot)$ is concave (equivalent to $u''(x) \leq 0$)

(iii) $c(F, u) \leq \int x \; dF(x)$



$P(\$1) = 0.5 \quad P(\$3) = 0.5$

Absolute Risk Aversion

Coefficient of Absolute Risk Aversion: $r_A(x, u) = \dfrac{-\,u''(x)}{u'(x)} \qquad > 0$

- Measures the curvature of $u(\cdot)$ at x

Proposition 6.5: The following statements are equivalent (comparison between individuals):

(i) $r_A(x, u_2) \geq r_A(x, u_1) \;\; \forall x$

(ii) There exists an increasing & concave function $\psi(\cdot)$ such that $u_2 = \psi(u_1(x)) \;\; \forall x$; u_2 is a concave transformation of u_1 (u_2 is "more concave" than u_1)

(iii) $c(F, u_2) \leq c(F, u_1) \;\; \forall F(\cdot)$

(iv) If u_2 finds $F(\cdot)$ at least as good as riskless outcome $\bar{x}$, then u_1 finds the same:
$U_2(F) = \int u_2(x) \; dF(x) \geq u_2(\bar{x}) \quad \Rightarrow \quad U_1(F) = \int u_1(x) \; dF(x) \geq u_1(\bar{x}) \quad \forall F(\cdot), \bar{x}$

$u(\cdot)$ exhibits **decreasing absolute risk aversion (DARA)** if $r_A(\cdot, u)$ is a decreasing function of x (comparison across wealth levels)

- If $F(\cdot)$ is preferred to z for some initial wealth level, then $F(\cdot)$ will be preferred to z for any higher initial wealth level
- $\int u(\tilde{x} + x) \; dF(\tilde{x}) > u(z + x) \quad \Rightarrow \quad \int u(\tilde{x} + x') \; dF(\tilde{x}) > u(z + x') \quad \forall x' > x$

Relative Risk Aversion

Given $x \geq 0$, a **proportionate gamble** pays tx, where t is some nonnegative random variable with cdf $F(\cdot)$

$$U(t, x) = \mathbb{E}[u(tx)] = \int u(tx) \; dF(t)$$

A **certain rate of return** $cr(F, x, u) = \widehat{t}$ is the rate of return for which the agent is indifferent between proportionate gamble tx and the certain rate

$$U(t, x) = \int u(tx) \; dF(t) = u(\widehat{t}x)$$

Coefficient of Relative Risk Aversion: $r_R(x,u) = \dfrac{-\,xu''(x)}{u'(x)} \quad > 0$

$r_R(x,u) = x\; r_A(x,u)$ Decreasing relative risk aversion $\Rightarrow$ decreasing absolute risk aversion
Increasing relative risk aversion $\Leftarrow$ increasing absolute risk aversion
$\forall x > 0$

Proposition 6.6: $r_R(\cdot, u)$ is a decreasing function of $x \;\Leftrightarrow\; cr(F,\cdot,u)$ is increasing in x

Comparing Risky Prospects

Assume that $F(0) = 0$ and $\exists x \ni F(x) = 1$

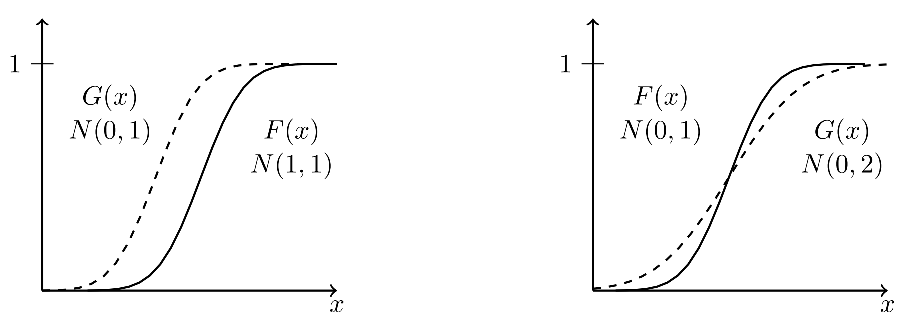
The distribution F **first order stochastically dominates** the distribution G (F FOSD G) iff, for every nondecreasing function $u : \mathbb{R} \to \mathbb{R}$, $\int u(x)\; dF(x) \geq \int u(x)\; dG(x)$

- F yields higher returns than G: $U(F) \geq U(G)\;\forall$ nondecreasing $u(\cdot)$
- F FOSD $G \;\Rightarrow\;$ the expected value of F is higher than that of G: $\int x\; dF(x) \geq \int x\; dG(x)$
- For any G, we can generate a distribution F that is weakly preferred by all agents with nondecreasing $u(\cdot)$, by shifting out the cdf (e.g., by improving one outcome with some probability, or adding a non-mean-zero random variable)

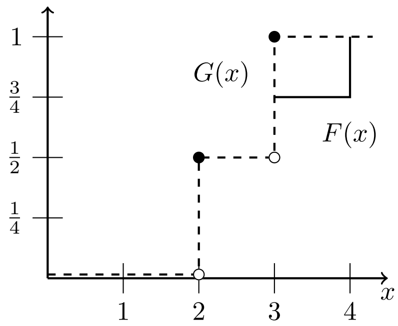
Proposition 6.7: F FOSD $G \;\Leftrightarrow\; F(x) \leq G(x)\;\forall x$

The distribution F **second order stochastically dominates** the distribution G (F SOSD G) if they have the same mean and, for every nondecreasing concave function $u : \mathbb{R} \to \mathbb{R}$, $\int u(x)\; dF(x) \geq \int u(x)\; dG(x)$

- F is less spread out (is less risky) than G: $U(F) \geq U(G)\;\forall$ nondecreasing, risk averse $u(\cdot)$
- If F and G have the same mean, F SOSD $G \;\Leftrightarrow\; \int_0^x G(t)\; dt \geq \int_0^x F(t)\; dt\;\forall x$
- For any F, we can create a distribution G that is weakly un-preferred by all agents with nondecreasing and risk averse $u(\cdot)$, by using a mean-preserving spread (e.g., by splitting one outcome into two equally likely outcomes $\pm\, y$, or adding a mean-zero random variable)



F FOSD G F SOSD G



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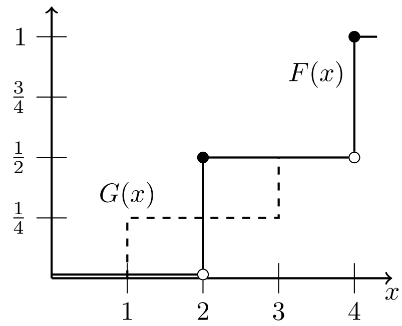


F SOSD G

Applications

When risk is actuarially favorable, then a risk averse agent will always accept at least a small amount of it.

Demand for Insurance

Consider a strictly risk averse agent with wealth w who risks losing $D \leq w$ with probability π. One unit of insurance costs q dollars and pays 1 dollar if loss occurs. α represents the number of insurance units purchased. The optimization problem is thus:

$$\max_{\alpha \geq 0} U = (1-\pi)u(w - \alpha q) + \pi u(w - \alpha q - D + \alpha)$$

Optimal choice α^* must satisfy the K-T FOC:

$$\frac{\partial U}{\partial \alpha^*} = -q(1-\pi)u'(w - \alpha^* q) + \pi(1-q)u'(w - D + \alpha^*(1-q)) \leq 0 \text{ with equality if } \alpha^* > 0$$

Strictly risk averse $\Leftrightarrow$ $u''(x) < 0$ $\Leftrightarrow$ $u'(x)$ is (strictly) decreasing

If insurance is actuarially fair, i.e., $q = \pi$, then the agent insures completely, i.e., $\alpha^* = D$. If insurance is not actuarially fair, i.e., $q > \pi$, assuming an interior solution, then the agent does not fully insure, i.e., $\alpha^* < D$.

- $\alpha^*(w)$ is decreasing in w if the agent exhibits DARA

Demand for a Risky Asset

Wealth w can be divided into α units of a risky asset (with random return of z dollars per dollar invested with cdf $F(\cdot)$ satisfying $\int z \, dF(z) > 1$), and $w - \alpha$ units of safe asset (with return of 1 dollar per dollar invested). The optimization problem is thus:

$$\max_{0 \leq \alpha \leq w} U = \int u(w + \alpha(z-1)) \, dF(z)$$

Optimal choice α^* must satisfy the K-T FOC:

$$\frac{\partial U}{\partial \alpha^*} = \phi(\alpha^*) = \int u'(w + \alpha^*(z-1))(z-1) \, dF(z) \begin{cases} \leq 0 & \text{if } \alpha^* < w \\ \geq 0 & \text{if } \alpha^* > 0 \end{cases}$$

Because $\phi(0) > 0$, we know that $\alpha^* > 0$ to satisfy this condition.

Appendix

Kuhn-Tucker

We can use Kuhn-Tucker for differentiable functions and convex (e.g., non-discrete) choice sets.

Consider a maximization problem with objective function $f(\mathbf{x})$ and K inequality constraints:

$$\max_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \quad s.t. \quad c_1 \geq g_1(\mathbf{x}) \ \ldots \ c_K \geq g_K(\mathbf{x})$$

Set up the Lagrangian:

$$\mathcal{L} = f(\mathbf{x}) + \lambda_1[c_1 - g_1(\mathbf{x})] + \cdots + \lambda_K[c_K - g_K(\mathbf{x})]$$

- Each constraint has a Lagrange multiplier
- The constraints are set up such that violating the constraint incurs a "penalty" − if the constraint is not met, the whole Lagrangian is decreased

Solution candidates satisfy:

(i) The FOC with respect to each choice variable equals zero

- $\frac{\partial \mathcal{L}}{\partial x_1} = 0 \quad \ldots \quad \frac{\partial \mathcal{L}}{\partial x_n} = 0$

(ii) The constraints

- $c_k \geq g_k(\mathbf{x}) \ \forall k$

(iii) Each multiplier times its constraint equals 0 (complementary slackness)

- $\lambda_k[c_k - g_k(\mathbf{x})] = 0 \ \forall k$
- Either the constraint binds (so $c_k - g_k(\mathbf{x}) = 0$), or it doesn't bind, in which case we might as well have done the problem without the constraint

(iv) Positive multipliers

- $\lambda_k \geq 0 \ \forall k$
- Supports the "penalty" for violating the constraint as above

Candidates can be found by investigating every potential combination of binding constraints, from no constraints bind to all constraints bind. When constraint $c_k \geq g_k(\mathbf{x})$ binds, $c_k = g_k(\mathbf{x})$; when it doesn't bind, $\lambda_k = 0$. Sometimes you can be sure a constraint binds without testing all combinations (e.g., equality constraint or budget constraint with an increasing utility function). Make sure to solve for all multipliers to make sure they are ≥ 0, and that constraints that are assumed to be binding/non-binding are actually so.

Kuhn-Tucker shows us that $\nabla f(\mathbf{x}) = \sum_{k=1}^{K} \lambda_k \nabla g_k(\mathbf{x})$. At the optimum, the gradient of the objective function is a linear combination of the gradient of the constraints that bind (the ones with multipliers > 0). So, at the optimum, λ_k gives the marginal value of relaxing the k^{th} constraint. For example, one marginal unit of wealth pushes the budget constraint out by one unit, allowing the agent to climb the utility curve by amount λ (get λ additional utility).

These four conditions are in general necessary but not sufficient for solutions. Thus if there is a solution, it will satisfy these conditions. If there are multiple candidates, plug them into $f(\mathbf{x})$ to see which returns the highest value.

If the objective function is quasiconcave in the choice variables, and the constraint set is convex in choice variables, then the conditions are necessary and sufficient (candidate $\Leftrightarrow$ solution). If the objective function is strictly quasiconcave, the solution is unique.

Non-negativity: Consider $\max_{\mathbf{x}\in\mathbb{R}^n} u(\mathbf{x}) \quad s.t. \quad w \geq p_1x_2 + \cdots + p_nx_n \quad x_i \geq 0 \ \forall i$

$$\mathcal{L} = u(\mathbf{x}) + \lambda[w - p_1x_1 - \cdots - p_nx_n]) + \mu_1[x_1 - 0] + \cdots + \mu_n[x_n - 0]$$

Our FOCs are $\dfrac{\partial u}{\partial x_i} = \lambda p_i - \mu_i \ \forall i$ and our complementary slackness conditions are $\mu_i x_i = 0 \ \forall i$

Because we know that $\mu_i \geq 0 \ \forall i$, we can rewrite the FOCs as $\dfrac{\partial u}{\partial x_i} \leq \lambda p_i$ with equality if $x_i > 0$

Envelope Theorem

The envelope theorem helps us understand how the value function of an optimization problem changes with respect to parameter changes. Consider an optimization problem with unique solution x^*, which generates a value function $V(\theta) = f(x^*(\theta), \theta)$, and has Lagrangian $\mathcal{L}(x, \theta, \lambda)$.

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{\partial \mathcal{L}}{\partial \theta}\bigg|_{\text{optimum}}$$

Roy's Identity: $\mathcal{L} = u(x_1, x_2) + \lambda(w - p_1x_1 - p_2x_2)$

$$V(p_1, p_2, w) = v(p_1, p_2, w) = u(x_1^*, x_2^*)$$

$$\frac{\partial v(p_1, p_2, w)}{\partial p_1} = \frac{\partial \mathcal{L}}{\partial p_1}\bigg|_{x_1^*, x_2^*} = -\lambda x_1^* \qquad \frac{\partial v(p_1, p_2, w)}{\partial w} = \frac{\partial \mathcal{L}}{\partial w}\bigg|_{x_1^*, x_2^*} = \lambda$$

$$x_1^* = \frac{-\,\partial v(p_1, p_2, w)/\partial p_1}{\partial v(p_1, p_2, w)/\partial w}$$

Shephard's Lemma: $\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(u(x_1, x_2) - u)$

$$V(p_1, p_2, u) = e(p_1, p_2, u) = p_1h_1^* + p_2h_2^*$$

$$\frac{\partial e(p_1, p_2, u)}{\partial p_1} = \frac{\partial \mathcal{L}}{\partial p_1}\bigg|_{h_1^*, h_2^*} = h_1^*$$

Marginal Rate of Substitution

To be on the same indifference curve, the change in utility due to the change in x_1 must be equal to the change in utility due to the change in x_2.

$\Delta u_{\text{good 1}} + \Delta u_{\text{good 2}} = 0$

$\dfrac{\partial u(x)}{\partial x_1}\, dx_1 + \dfrac{\partial u(x)}{\partial x_2}\, dx_2 = 0$

$\text{Slope} = \dfrac{dx_2}{dx_1} = \dfrac{-\,\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} = -MRS_{12}$

$MRS_{12} = \dfrac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} \quad \approx \quad \dfrac{-\,\Delta x_2}{\Delta x_1}$ The additional amount of x_2 needed to compensate for giving up a marginal unit of x_1

Optimally, the slope of the indifference curve must match the slope of the budget line: $MRS_{12} = \dfrac{p_1}{p_2}$

Notation

$x \geq y$	$x_n \geq y_n \;\; \forall n = 1, ..., N$
$x \gg y$	$x_n > y_n \;\; \forall n = 1, ..., N$

$x \in B$	Element x is included in set B
$A \subseteq B$	Set A is a subset of set B
$\{x\} \subseteq B$	The set that has x as its only element is included in set B

$$\nabla f(\mathbf{x}) = \left[\frac{\partial f(\mathbf{x})}{\partial x_1} \quad \frac{\partial f(\mathbf{x})}{\partial x_2} \quad \cdots \quad \frac{\partial f(\mathbf{x})}{\partial x_n} \right]$$

Real Analysis

Convex set: Set $A \subseteq \mathbb{R}^n$ is convex if $\alpha x + (1-\alpha) y \in A \;\; \forall x, y \in A, \alpha \in [0, 1]$

- The intersection of two convex sets is convex

Closed: Set D is closed if for every sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in D$ and $x_n \rightarrow x$, it is also the case that $x \in D$ (set D includes its boundaries)

Bounded: A set is bounded if you can draw a circle around it

Compact: A set is compact if it is closed and bounded

Concave function: A function $f(\cdot)$ is concave if $f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha) f(x') \;\; \forall \alpha \in [0,1]$

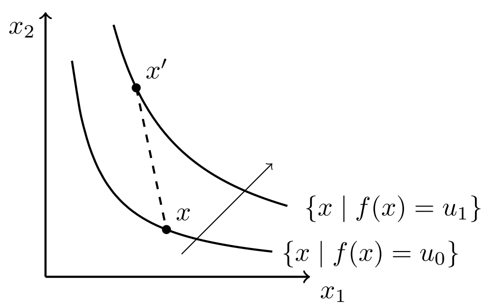
- Strictly concave if $<$ and $\alpha \in (0,1) \;\; \forall x \neq x'$
- $f(\cdot)$ is a concave function $\;\Leftrightarrow\;$ the set on and below $f(\cdot)$ is a convex set
- $f(\cdot)$ is a concave function $\;\Leftrightarrow\;$ $f(\mathbb{E}[x]) \geq \mathbb{E}[f(x)]$ (Jensen's Inequality)

Convex function: A function $f(\cdot)$ is convex if $f(\alpha x + (1-\alpha)x') \leq \alpha f(x) + (1-\alpha) f(x') \;\; \forall \alpha \in [0,1]$

- Strictly convex if $>$ and $\alpha \in (0,1) \;\; \forall x \neq x'$
- $f(\cdot)$ is a convex function $\;\Leftrightarrow\;$ the set on and above $f(\cdot)$ is a convex set
- $f(\cdot)$ is a convex function $\;\Leftrightarrow\;$ $f(\mathbb{E}[x]) \leq \mathbb{E}[f(x)]$ (Jensen's Inequality)

Quasiconcave function: A function $f(\cdot)$ is quasiconcave if $f(\alpha x + (1-\alpha)x') \geq \min\{f(x), f(x')\} \;\; \forall \alpha \in [0,1]$

- Strictly quasiconcave if $>$ and $\alpha \in (0,1) \;\; \forall x \neq x'$
- $f(\cdot)$ is a quasiconcave function $\;\Leftrightarrow\;$ the superior set $\{y \mid f(y) \geq f(x)\}$ is a convex set $\;\forall x$
- $f(\cdot)$ is a strictly quasiconcave fcn $\;\Leftrightarrow\;$ the superior set $\{y \mid f(y) \geq f(x)\}$ is a strictly convex set $\;\forall x$



Quasiconvex function: A function $f(\cdot)$ is quasiconvex if $f(\alpha x + (1-\alpha)x') \leq \max\{f(x), f(x')\} \;\; \forall \alpha \in [0,1]$

- Strictly quasiconvex if $<$ and $\alpha \in (0,1) \;\; \forall x \neq x'$
- $f(\cdot)$ is a quasiconcave function $\;\Leftrightarrow\;$ the inferior set $\{y \mid f(y) \leq f(x)\}$ is a convex set $\;\forall x$
- $f(\cdot)$ is a strictly quasiconcave fcn $\;\Leftrightarrow\;$ the inferior set $\{y \mid f(y) \leq f(x)\}$ is a strictly convex set $\;\forall x$

Logic

DeMorgan's Laws: $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$
$\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$

Conditional: $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$

Biconditional: $P \Leftrightarrow Q$ is equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Contrapositive: $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$

OR in the hypothesis: $P \Rightarrow Q \vee R$ is equivalent to $P \wedge \neg Q \Rightarrow R$

Hypothesis in the conclusion: $P \Rightarrow (Q \Rightarrow R)$ is equivalent to $(P \wedge Q) \Rightarrow R$

Proof by Mathematical Induction: Want to show $P(n)$ is true $\forall n \in N$

- Show the base case is true (e.g, $P(0)$ or $P(1)$)
- Make inductive hypothesis: "Assume $P(n)$ is true"
- Show $P(n+1)$ must be true

To Prove:

- $A \subseteq B$: Start with arbitrary $x \in A$ and show $x \in B$
- $A = B$: Show $A \subseteq B$ and $B \subseteq A$
- $A \subset B$: Show $A \subseteq B$ and find any $y \in B$ such that $y \notin A$ [$\neg(B \subseteq A)$]
- $A \cap B = \emptyset$: Suppose towards contradiction that arbitrary $x \in A \cap B$ and show a contradiction