Interest Rates

Many of our formulas incorporate an interest rate r. For example, if we want to calculate the present value of C dollars received in period T with interest rate r:

$$PV_0 = \frac{C_T}{(1+r)^T}$$

However, we need to think carefully about which r to use. **How we count time (how we define a period) must match with our choice of r.**

Definitions (using my preferred notation)

$r_{APR,m}$: Stated annual interest rate (or annual percentage rate)

- The stated rate is always associated with a compounding frequency, m
- $m = \#$ of compounding periods per year
- I put m in the subscript so I don't forget to keep track of the compounding frequency

$\frac{r_{APR,m}}{m}$: Effective per-period rate

- The stated rate is a complicated way of saying the bank pays $\frac{r_{APR,m}}{m}$ every period m times per year

r_{EAR} : Effective annual rate (or annual percentage yield)

- The effective rate is the interest rate compounding once per year that is equivalent to $r_{APR,m}$

Converting between different interest rates

Suppose a bank pays a stated rate compounded quarterly, $r_{APR,4}$. Think about investing \$1 for one year. The end balance is:

$$\$1\underbrace{\left(1+\frac{r_{APR,4}}{4}\right)}_{\text{1st quarter}}\underbrace{\left(1+\frac{r_{APR,4}}{4}\right)}_{\text{2nd quarter}}\underbrace{\left(1+\frac{r_{APR,4}}{4}\right)}_{\text{3rd quarter}}\underbrace{\left(1+\frac{r_{APR,4}}{4}\right)}_{\text{4th quarter}} = (1+r_{EAR})$$

where the r_{EAR} is the interest rate compounded once that results in the same end value.

We can convert between a stated rate (compounded m times per year) and the effective annual rate using:

$$\left(1+\frac{r_{APR,m}}{m}\right)^m = (1+r_{EAR})$$

Suppose that we want to convert between a stated rate compounded m times per year, and a stated rate compounded k times per year (e.g., between a stated daily and a stated monthly rate). We can use:

$$\left(1 + \frac{r_{APR,m}}{m}\right)^m = \left(1 + \frac{r_{APR,k}}{k}\right)^k$$

because both sides are equal to $(1 + r_{EAR})$.

Finally, for continuous compounding, we can write this as $m = \infty$, and use:

$$(1 + r_{EAR}) = e^{r_{APR,\infty}}$$

In summary, we can convert between interest rates using any combination of these four pieces:

$$\boxed{(1 + r_{EAR}) = \left(1 + \frac{r_{APR,m}}{m}\right)^m = \left(1 + \frac{r_{APR,k}}{k}\right)^k = e^{r_{APR,\infty}}}$$

where m and k are the number of compounding periods per year.

Which r to use?

How we define a period must match with our choice of r.

- If we choose period = year, we want to use $r = r_{EAR}$
- If we choose period = quarter, we want to use use $r = \frac{r_{APR,4}}{4}$
- If we choose period = month, we want to use $\frac{r_{APR,12}}{12}$
- …etc.

Example: MT1 Winter 2016 #2

Joanna will receive a single payment of \$10,000 7.5 years from today. You find out that her stated annual discount rate is 4.8%, compounded 24 times per year. What is the present value of this payment?

$C_T = 10000$
$r_{APR,24} = 4.8\%$

There are two ways to solve this problem:

Method 1) period $= \frac{1}{2}$ month

$T = 7.5 \cdot 24 = 180$ *(counting time in half months)*

Want $r = \frac{r_{APR,24}}{24} = \frac{0.048}{24} = 0.002$ *(the interest that accumulates in half a month)*

$$PV_0 = \frac{C_T}{(1+r)^T} = \frac{C_{180}}{(1+0.002)^{180}} = \frac{10000}{1.002^{180}} = 6979.27$$

Method 2) period = 1 year

$T = 7.5$
Want $r = r_{EAR}$ *(the interest that accumulates in one year)*

$$\begin{aligned}(1 + r_{EAR}) &= \left(1 + \frac{r_{APR,m}}{m}\right)^m \\ &= \left(1 + \frac{0.048}{24}\right)^{24} \\ &= 1.002^{24} \\ &= 1.0491\end{aligned}$$

$$PV_0 = \frac{C_T}{(1+r)^T} = \frac{10000}{1.0491^{7.5}} = 6979.27$$

This example is used to illustrate how the way periods are defined must match with our choice of r. In problems like this one, it doesn't matter what period we choose so long as we have an r that matches.

However, for perpetuities and annuities, we do not have a choice of how to define periods. The perpetuity and annuity formulas use the time in between payments as the period. As a result, we will need to find the r that matches (if it is not provided in the question).