Math Camp 2020 - Logic (Reference)*

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1. **Propositions and Connectives**

   (a) Definition. A **proposition** is a statement that has exactly one truth value: either true (denoted T) or false (denoted F).

   (b) Examples.

      i. "$1 + 1 = 2$"

      ii. "Sarah won a Nobel Prize in economics in 2019"

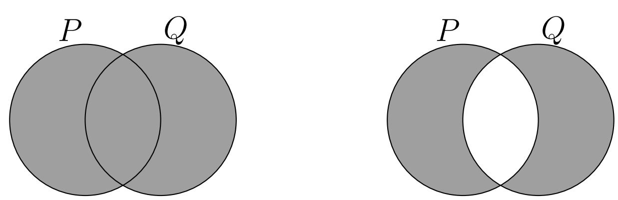
      iii. "$x^2 = 36$" (not a proposition because it is true for some x and false for others)

   (c) Definition. The **negation** of the proposition P, denoted $\neg P$, is the proposition "not P." $\neg P$ is true when P is false.

   (d) Definition. Given propositions P and Q, the **conjunction** of P and Q, denoted $P \wedge Q$, is the proposition "P and Q." $P \wedge Q$ is true when both P is true and Q is true.

   (e) Definition. Given propositions P and Q, the **disjunction** of P and Q, denoted $P \vee Q$ is the proposition "P or Q." $P \vee Q$ is true when P is true or Q is true.

   (f) Aside. This is known as the "inclusive or," e.g., "P or Q or both P and Q." There is also an "exclusive or," e.g., "P or Q but not both P and Q." Presented as Venn Diagrams:



Inclusive Or
$P \vee Q$

Exclusive Or
$(P \vee Q) \wedge \neg(P \wedge Q)$

   (g) Example. For the different combinations of truth values for the propositions P and Q, we can determine the truth values of our various more complicated compound propositions:

*These notes are drawn principally from *A Transition to Advanced Mathematics*, 7th ed., by Douglas Smith, Maurice Eggen, and Richard St. Andre, and *How To Prove It: A Structured Approach*, 2nd ed., by Daniel Velleman. The material posted on this website is for personal use only and is not intended for reproduction, distribution, or citation. James Banovetz created the first edition of these awesome notes and graciously shared them.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$P \vee Q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	F	F

(h) Aside. This is known as a **truth table**. It is a helpful tool that lets us organize the truth value of propositions and connectives. You won't use them during the first year coursework, but it helps us get a sense for the form of logical statements and proofs.

(i) Definition. A **tautology** is a propositional form that is true for every assignment of truth values to its components.

(j) Example. Consider the propositional form $P \vee \neg P$:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

(k) Definition. A **contradiction** is a propositional form that is false for every assignment of truth values to its components.

(l) Example. Consider the propositional form $P \wedge \neg P$:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

(m) Aside. While these are trivial examples of tautologies and contradictions, they are important for proofs. "If and only if" statements are tautologies (which we frequently are trying to prove), while contradictions are a powerful tool for proving certain propositions.

(n) Definition. Two propositional forms are **equivalent** if they have the same truth tables.

(o) Example. P and $\neg(\neg P)$ are equivalent (draw a truth table for convincing!).

(p) Theorem (SES THM 1.1.1). The following propositional forms are equivalent:

i. Double Negation:
   - P and $\neg(\neg P)$

ii. Commutative Laws
   - $P \vee Q$ and $Q \vee P$
   - $P \wedge Q$ and $Q \wedge P$

iii. Associative Laws
   - $P \vee (Q \vee R)$ and $(P \vee Q) \vee R$
   - $P \wedge (Q \wedge R)$ and $(P \wedge Q) \wedge R$

iv. Distributive Laws
   - $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$
   - $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$

v. Idempotent Laws
- $P \lor P$ and P
- $P \land P$ and P

vi. Absorption Laws
- $P \lor (P \land Q)$ and P
- $P \land (P \lor Q)$ and P

vii. DeMorgan's Laws
- $\neg(P \land Q)$ and $\neg P \lor \neg Q$
- $\neg(P \lor Q)$ and $\neg P \land \neg Q$

(q) Aside. These are important in formulating proofs, particularly in first quarter microeconomics. For example, imagine we want to prove the statement "preferences are not complete and transitive" ($\neg(P \land Q)$). This is equivalent to proving that "preferences are not complete or preferences are not transitive" ($\neg P \lor \neg Q$). Moving between equivalent propositional forms is important, because it provides insight into what we need to accomplish with the proof. In this case, we can show $\neg P$ or $\neg Q$.

2. **Conditionals and Biconditionals**

(a) Definition. For propositions P and Q, the **conditional sentence** $P \implies Q$ is the proposition "if P, then Q." The conditional sentence $P \implies Q$ is true exactly when P is false or Q is true. The truth table associated with $P \implies Q$ is:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

(b) Example. We can think of $P \implies Q$ in terms of a promise. For example, consider the promise, IF I draw [*name*]'s name, THEN I will pay [*name*] \$10. Then $P \equiv$ "I draw [*name*]'s name" and $Q \equiv$ "I pay [*name*] \$10." Suppose I draw [*name*]:

i. P is True

ii. If I pay, then Q is True
- I didn't break the promise ($P \implies Q$ is True)

iii. If I don't pay, then Q is False
- I broke the promise ($P \implies Q$ is False)

On the other hand, suppose I DON'T draw [*name*]:

i. P is False

ii. If I pay, then Q is True
- I didn't break the promise ($P \implies Q$ is True)

iii. If I don't pay, then Q is False
- I didn't break the promise ($P \implies Q$ is True)

(c) Aside. The conditional statement is the most important propositional form in mathematics and economics. Every "if then" and "implies" statement is a conditional sentence. "IF the minimum wage is binding, THEN there will be involuntary unemployment." "The existence of a utility function IMPLIES complete, transitive preferences." Note that in common English $P \implies Q$ contains casual connotations, but this is not the case in mathematics.

(d) Aside. We might also say the conditional as "P is a sufficient condition for Q".

(e) Aside. In order to prove $P \implies Q$, we are going to start by assuming P is true and showing Q is true (we will show the first line from the truth table holds, and that the second line does not hold).

If P is false, then $P \implies Q$ is automatically true for any Q. Consider these two propositions:

$$\text{Dick Startz was the first man on the Moon} \implies \Big(1 + 1 = 2\Big)$$

$$\text{Dick Startz was the first man on the Moon} \implies \Big(1 + 1 = 500\Big)$$

Both of these propositions are true, because the P part is false. Whether Q is true or not is irrelevant. We don't worry about the line third and fourth lines of this truth table when we are writing proofs (we get them for free).

(f) Definition. For propositions P and Q, the **converse** of $P \implies Q$ is $Q \implies P$. The **contrapositive** of $P \implies Q$ is $\neg Q \implies \neg P$.

(g) Theorem. (SES THM 1.2.1) The proposition $P \implies Q$ is equivalent to its contrapositive $\neg Q \implies \neg P$. It is *not* equivalent to its converse $Q \implies P$.

Expanding the truth table for the conditional $P \implies Q$:

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$	$Q \Rightarrow P$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	F
F	F	T	T	T	T	T

(h) Aside. This is an incredibly useful theorem, despite the fact that it may not be intuitive at first glance. As mentioned above, moving between equivalent propositional forms can be helpful when we're writing proofs; few (if any) are as frequently used as the contrapositive.

(i) Definition. For propositions P and Q, the **biconditional sentence** $P \iff Q$ is the proposition "P if and only if Q." $P \iff Q$ is true exactly when P and Q have the same truth values.

The truth table associated with $P \iff Q$ is:

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

(j) Aside. Note that we frequently write "if and only if" as iff. All (properly stated) definitions are biconditionals, as they lay out the exact conditions to meet the definition.

(k) Aside. We might also say the biconditional as "P is a necessary and sufficient condition for Q."

(l) Theorem (SES THM 1.2.2) For propositions P, Q, and R:

   i. $P \implies Q$ is equivalent to $(\neg P) \vee Q$ (definition of conditional)

   ii. $P \iff Q$ is equivalent to $(P \implies Q) \wedge (Q \implies P)$ (definition of biconditional)

   iii. $P \implies (Q \vee R)$ is equivalent to $(P \wedge \neg R) \implies Q$ and to $(P \wedge \neg Q) \implies R$ (or in the hypothesis)

   iv. $P \implies (Q \implies R)$ is equivalent to $(P \wedge Q) \implies R)$ (hypothesis in the conclusion)

   v. $P \implies (Q \wedge R)$ is equivalent to $(P \implies Q) \wedge (P \implies R)$

   vi. $(P \vee Q) \implies R$ is equivalent to $(P \implies R) \wedge (Q \implies R)$

(m) Aside. If you can't remember the name of a logical equivalence you want to use, "logic" can be an acceptable justification for a step in a proof IF it's obviously equivalent to whoever is grading it (they are sure you aren't making it up). So use your best judgment. If it's not obvious in one step, try breaking it down into smaller steps.

3. **Quantifiers**

(a) Aside. Recall that when defining propositions, we saw that $x^2 = 36$ was not a proposition, because it is true for some x and false for others. However, we might want to say something about how many values of x make $x^2 = 36$ true. Quantifiers help us express these ideas.

(b) Definition. A statement that contains a variable is called an **open sentence**. It becomes a proposition only when its variables are assigned specific values.

(c) Definition. Given an open sentence, the collection of permissible objects available for consideration is the **universe of discourse**.

(d) Definition. For an open sentence $P(x)$ and universe U, the sentence $\exists\ x \in U \ni P(x)$ reads "there exists an x in U such that $P(x)$." The symbol $\exists$ is called the **existential quantifier**.

(e) Definition. For an open sentence $P(x)$ and universe U, the sentence $\exists!\ x \in U \ni P(x)$ reads "there exists exactly one x in U such that $P(x)$." The symbol $\exists!$ is called the **uniqueness existential quantifier**.

(f) Definition. For an open sentence $Q(x)$, the sentence $\forall\ x \in U, P(x)$ reads "for all x in U, $P(x)$." The symbol $\forall$ is called the **universal quantifier**.

(g) Example. The following are propositions (where $\mathbb{R}$ is the set of real numbers):

   i. $\exists\ x \in \mathbb{R} \ni x^2 = 36$ (true)

   ii. $\exists!\ x \in \mathbb{R} \ni x^2 = 36$ (false)

   iii. $\forall\ x \in \mathbb{R},\ x^2 = 36$ (false)

(h) Aside. Essentially, $\exists$ and $\forall$ take open sentences and make them propositions. $\exists$ means that there is at least one value that makes an open sentence true ($\exists!$ means there is exactly one); $\forall$ says the open sentence is true for every value. You will use these quantifiers a lot during the first year classes, so make sure you're comfortable with them.

(i) Aside. In logic and mathematics, specifying what universe you're considering can be quite important. In particular, you must be extremely careful in Econ 241A–you'll be docked points if you don't specify the relevant universes (known as supports) for distributions. If the universe is already completely clear, you can leave it out (some are dropped below to make things easier to read).

(j) <u>Theorem</u> (SES THM 1.3.1). If $P(x)$ is an open sentence with variable x in universe U, then

i. $\neg(\forall\, x, P(x))$ is equivalent to $\exists\, x \ni \neg P(x)$

ii. $\neg(\exists\, x \ni P(x))$ is equivalent to $\forall\, x, \neg P(x)$

(k) <u>Example</u>.

- Our universe of discourse is "all social scientists"
- $Q(y) \equiv$ "bad at math"

Consider the statement:

$$\neg\Big[\forall y, \quad Q(y)\Big]$$

This statement reads, "not, for all social scientists y, y is bad at math," or, in plain English, "it is not true that all social scientists are bad at math."

What would it take for the statement "all social scientists are bad at math" to be false? At least one that is good at math! In other words: "there exists a social scientist who is good at math."

$$\exists\, y \ni Q(y)$$

(l) <u>Example</u>. Suppose you have an exam question: "Weakly dominated strategies cannot be part of Nash Equilibria. True or False. Provide a proof."

Knowing that $\forall$ and $\exists$ are the negations of each other is extremely important for proofs. In this example, the statement is false. Further, All that is required of the proof is to provide ONE example where a weakly dominated strategy is part of a NE.

(m) <u>Aside</u>. This gets at an important point: to disprove a "for all" proposition, we just need one counterexample to prove it's false. This type of question is very common for the first-quarter micro sequence!

(n) <u>Example</u>. Let $\mathbb{N}$ be the "natural numbers" (i.e., 1, 2, 3, ...). Find the negation of the proposition

$$\begin{aligned}
&\exists\, x \in \mathbb{N} \ni x < 2 \wedge x \neq 1 && \text{(a false propostion)}\\
&\neg\,(\exists\, x \in \mathbb{N} \ni x < 2 \wedge x \neq 1) && \text{(the negation)}\\
&\forall\, x \in \mathbb{N}, \neg(x < 2 \wedge x \neq 1) && \text{(by SES THM 1.3.1)}\\
&\forall\, x \in \mathbb{N}, x \geq 2 \vee x = 1 && \text{(by DeMorgan's Laws)}
\end{aligned}$$

This is a true proposition!

4. **Basics of Writing Proofs**.

(a) <u>Definition</u>. Initial sets of statements assumed to be true are called **axioms**.

(b) <u>Outline of Proof Writing</u>.

i. List definitions, axioms, previously proved results/theorems, or assumptions (be careful about assumptions).

ii. At any time, replace statements with equivalent statements

iii. At any time, state tautologies

(c) <u>Form of Direct Proofs</u>. Suppose we're trying to prove that $P \implies Q$. direct proofs frequently look something like:

- List relevant definitions, axioms, theorems, assumptions, etc.

- State "Direct proof to show Q" (or equivalent statement)
- Proof:

$$\begin{array}{ll} \text{Let } P \text{ be true} & \text{(by hypothesis)} \\ \text{Then } \textit{replace} & \text{(by results/theorems)} \\ \text{Consider } \textit{tautology} & \text{(by tautology rule)} \\ \vdots & \\ \text{Then } Q & \end{array}$$

Thus, $P \implies Q$.

(d) Example. Let x be an integer. Prove that if x is odd, then $x + 1$ is even.

- $\mathbb{Z}$ is the set of integers
- Def. of even: $y \in \mathbb{Z}$ is even $\iff \exists\, k \in \mathbb{Z} \ni y = 2k$
- Def. of odd: $x \in \mathbb{Z}$ is odd $\iff \exists\, j \in \mathbb{Z} \ni x = 2j + 1$
- Closure property: the sum of two integers is an integer
- Successor property: If $x \in \mathbb{Z}$, x has a unique successor $x + 1$

Direct proof to show: $x + 1$ is even.
Proof:

$$\begin{array}{ll} \text{Let } x \text{ be an odd integer} & \text{(by hypothesis)} \\ \implies \exists\, k \in \mathbb{Z} \ni x = 2k + 1 & \text{(by def. of odd)} \\ \implies x + 1 = (2k + 1) + 1 & \text{(by succession/closure)} \\ \implies x + 1 = 2k + 2 & \text{(by associativity)} \\ \implies x + 1 = 2(k + 1) & \text{(by distributivity)} \\ \implies (k + 1) \text{is an integer} & \text{(by closure)} \\ \implies x + 1 \text{ is even} & \text{(by def. of even)} \end{array}$$

$\blacksquare$

(e) Form of Proof by Controposition. Suppose we're trying to prove that $P \implies Q$. We can prove the contrapositive instead, as it is an equivalent statement:

- List relevant definitions, axioms, theorems, assumptions, etc.
- State the contrapositive
- State "Proof by contraposition to show $\neg P$"
- Proof:

$$\begin{array}{ll} \text{Let } \neg Q \text{ be true} & \text{(by hypothesis)} \\ \text{Then } \textit{replace} & \text{(by results/theorems)} \\ \text{Consider } \textit{tautology} & \text{(by tautology rule)} \\ \vdots & \\ \text{Then } \neg P & \end{array}$$

Thus, $P \implies Q$.

(f) Example. Let m be an integer. Prove that if m^2 is even, then m is even.

- Contrapositive: if m is not even, then m^2 is not even
- Assumption 1: An integer is odd if and only if it is not even

Proof by contraposition to show: m^2 is not even
Proof:

$$\begin{aligned}
&\text{Let } m \text{ be a not even integer} && \text{(by hypothesis)}\\
&\implies m \text{ is odd} && \text{(by Assumption 1)}\\
&\implies \exists\, k \in \mathbb{Z} \ni m = 2k+1 && \text{(by def. of odd)}\\
&\implies m^2 = (2k+1)^2 && \text{(squaring both sides)}\\
&\implies m^2 = 4k^2 + 4k + 1 && \text{(expanding)}\\
&\implies m^2 = 2(2k^2 + 2k) + 1 && \text{(rearranging)}\\
&\implies 2k^2 + 2k \text{ is an integer} && \text{(by closure)}\\
&\implies m^2 \text{ is odd} && \text{(by def. of odd)}\\
&\implies m^2 \text{ is not even} && \text{(by Assumption 1)}
\end{aligned}$$

■

Thus, we have proved the contrapositive of "if m^2 is even, then m is even." Since the contrapositive is equivalent to the original statement, this is a sufficient proof.

(g) Aside. Note a few points: first, we assumed that an integer is even if and only if it isn't odd. We could have proved this, but you always need to take some things as given. Second, we assumed that the usual rules of algebra held (I didn't explictitly write down all the steps and associated assumptions).

When you're actually writing proofs during the first year, you will end up assuming quite a few things for convenience–just be careful you don't assume away the proof! Typically after the first week of so, you won't need to state things that seem obvious to us, like the rules of algebra (associativity, etc.) or that $0 \cdot a = 0$. Again, DO NOT make assumptions that make the proof trivially easy!

(h) Form of Proof by Contradiction. Proof by contradiction is a powerful tool. First, "suppose towards contradiction" something that you want to reject (e.g., $\neg R$). Then, show a contradiction (e.g., $Q \wedge \neg Q$). Finally, you can reject the thing that you supposed (e.g., conclude R).

Say you want to prove $(P \wedge Q) \Rightarrow R$:

- List relevant definitions, axioms, theorems, assumptions, etc.
- State "Proof by contradiction to show R"
- Proof:

$$\begin{aligned}
&\text{Let } P \wedge Q && \text{(by hypothesis)}\\
&\text{Suppose } \neg R && \text{(towards a contradiction)}\\
&\vdots && \\
&\text{Then } \neg Q && \\
&\text{Then } Q \wedge \neg Q && \\
&\text{Then } R && \text{(by contradiction)}
\end{aligned}$$

(i) Example. Let $x^2 + y = 13$ and $y \neq 4$. Prove that $x \neq 3$.

To Show: $x \neq 3$
Proof:

$$
\begin{array}{lr}
\text{Let } x^2 + y = 13 & \text{(by hypothesis)} \\
\text{Let } y \neq 4 & \text{(by hypothesis)} \\
\text{Suppose } x = 3 & \text{(towards contradiction)} \\
\implies (3)^2 + y = 13 & \text{(algebra)} \\
\implies y = 4 & \text{(algebra)} \\
\implies y \neq 4 \text{ and } y = 4 & \text{(logic)} \\
\implies x \neq 3 & \text{(by contradiction)}
\end{array}
$$

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5. **Applications**

(a) Biconditional Proofs. Suppose we're trying to prove $P \iff Q$. There are two ways to approach the problem: in two parts, or biconditionally.

i. Two-Part Proof (from SES THM 1.2.2)
   - Show $P \implies Q$
   - Show $Q \implies P$
   - Thus, $P \iff Q$

ii. Biconditional Proof
   - Every line must involve a biconditional:

$$
\begin{array}{lr}
\text{Let } P \text{ be true} & \text{(by hypothesis)} \\
\iff R_1 & \text{(by results/theorems/definitions)} \\
\vdots & \\
\iff R_n & \text{(by results/theorems/definitions)} \\
\iff Q &
\end{array}
$$

Example: Show that $\neg(P \wedge Q)$ if and only if $Q \Rightarrow \neg P$
To Show ($\Rightarrow$): $Q \Rightarrow \neg P$
Proof:

$$
\begin{array}{lr}
\text{Let } \neg(P \wedge Q) & \text{(by hypothesis)} \\
\implies \neg P \vee \neg Q & \text{(DeMorgan's Laws)} \\
\implies P \Rightarrow \neg Q & \text{(def. of conditional)} \\
\implies Q \Rightarrow \neg P & \text{(contrapositive)}
\end{array}
$$

■

To Show ($\Leftarrow$): $\neg(P \wedge Q)$

Proof:

$$\begin{aligned} &\text{Let } Q \Rightarrow \neg P && \text{(by hypothesis)} \\ \Longrightarrow\ & P \Rightarrow \neg Q && \text{(contrapositive)} \\ \Longrightarrow\ & \neg P \vee \neg Q && \text{(def. of conditional)} \\ \Longrightarrow\ & \neg(P \wedge Q) && \text{(DeMorgan's Laws)} \end{aligned}$$

■

In this case, we can do it on one proof because all of the steps use biconditional relationships (definitions).
To Show: $[\neg(P \wedge Q)] \iff [Q \Rightarrow \neg P]$
Proof:

$$\begin{aligned} &\text{Let } Q \Rightarrow \neg P && \text{(by hypothesis)} \\ \iff\ & P \Rightarrow \neg Q && \text{(contrapositive)} \\ \iff\ & \neg P \vee \neg Q && \text{(def. of conditional)} \\ \iff\ & \neg(P \wedge Q) && \text{(DeMorgan's Laws)} \end{aligned}$$

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(b) Proofs with Quantifiers Suppose we're trying to prove a "for all" or a "there exists" proposition $P(x)$. Depending on the quantifier, there are two basic approaches:

i. For-All Proofs
   - Pick an arbitrary element in the universe, $x \in U$
   - Show that x makes $P(x)$ true
   - Because x was arbitrary, it's true for all $x \in U$
   - Don't put any other restrictions on x after you've selected it, because then it isn't arbitrary anymore

ii. There-Exists Proofs
   - Pick a specific element in that universe, $x \in U$
   - Show that x is in U and makes $P(x)$ true

iii. There-Exists-Unique Proofs
   - Pick a specific element in that universe, $x \in U$
   - Show that x is in U and makes $P(x)$ true
   - Pick an arbitrary y that is in U and satisfies $P(x)$
   - Show that y must equal x

(c) Proof by Mathematical Induction
Example: For every $n \in \{0, 1, 2, 3, \dots\}$, prove that $2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1$
To Show: Base case $P(0)$

Proof:

$$
\begin{aligned}
&\text{Let } n = 0 && \text{(by hypothesis)}\\
&\implies 2^0 = 1 && \text{(algebra)}\\
&\implies 2^{n+1} - 1 = 1 && \text{(algebra)}\\
&\implies 2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1 && \text{(algebra)}
\end{aligned}
$$

To Show: $P(n) \Rightarrow P(n+1)$

Proof:

$$
\begin{aligned}
&\text{Assume } 2^0 + 2^1 + \cdots + 2^n = 2^{n+1} - 1 && \text{(by inductive hypothesis)}\\
&\implies 2^0 + 2^1 + \cdots + 2^n + 2^{n+1} = 2^{n+1} - 1 + 2^{n+1} && \text{(adding } 2^{n+1} \text{ to both sides)}\\
&\implies \quad \text{''} = 2(2^{n+1}) - 1 && \text{(algebra)}\\
&\implies \quad \text{''} = 2^{n+1+1} - 1 && \text{(algebra)}
\end{aligned}
$$

$\blacksquare$