Notation[1]

Pay attention to what's an *element* versus a *set* (group of elements).

x	An element (scalar, ordered pair, etc.)
$\{x\}$	A set that contains x as its only element (single-valued set)
$\{x, y, z\}$	A set containing only the elements x, y, z (multi-valued set)

A	A set
$x \in A$	Element x is included in set A
$x, y \notin A$	x and y are not included in A

$\mid$	"such that" (same as $\ni$ but what we use for sets)
$A = \{x \mid P(x)\}$	A is the set of all x such that $P(x)$

Example: $P(x) =$ "x is an odd integer between 0 and 6"
$A = \{x \mid P(x)\} = \{1, 3, 5\}$

$A \subseteq B$	Set A is a subset of set B
$A = B$	Set A equals set B
$A \subset B$	Set A is a strict subset of set B
$\{x\} \subseteq A$	The set that has x as its only element is a subset of set A (another way of saying that x is included in A)

[1]Prepared by Sarah Robinson

Common Sets

$\mathbb{N}$	Set of natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	Set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{R}$	Set of real numbers (whole number line)
$\mathbb{R}^+$ or $\mathbb{R}_+$	Set of non-negative real numbers
$\emptyset$	The empty set or null set (it is a subset of every set)
(a, b)	The open interval between real numbers a and b
$[a, b]$	The closed interval between real numbers a and b

$$(a, b) = \{x \mid x \in \mathbb{R} \wedge a < x < b\}$$
$$[a, b] = \{x \mid x \in \mathbb{R} \wedge a \leq x \leq b\}$$

$A \subseteq B$ iff every element of A is also an element of B.

We can write this using logic:
$A \subseteq B \iff \forall\, x, (x \in A \Rightarrow x \in B)$

If you want to prove that $A \subseteq B$, pick an arbitrary element $x \in A$ and show that $x \in B$.

$A = B \iff (A \subseteq B) \land (B \subseteq A)$
Every element in A is in B and vice versa (they are the same set).

If you want to prove that $A = B$, how would you do it?

$A \subset B \iff (A \subseteq B) \land \neg(B \subseteq A)$
$(B \not\subseteq A)$
There is at least one element in B that is not in A.

If you want to prove that $A \subset B$, first show that $A \subseteq B$ and then show that $B \not\subseteq A$ using a counter example.

Example:

$$A = \{x \mid a < x < b\} \qquad B = \{x \mid a \leq x \leq b\}$$

Prove that $A \subset B$.

<u>To Show:</u> $x \in B$

<u>Proof:</u>

Let $x \in A$	(by hypothesis)
$\implies a < x < b$	( )
$\implies a \leq x \leq b$	( )
$\implies x \in B$	( )

<u>To Show:</u> $\exists\ x \in B \ni x \notin A$

<u>Proof:</u>

Let $x = b$	(by hypothesis)
$\implies x \in B$	( )
$\implies \neg(x < b)$	( )
$\implies x \notin A$	( )

■

The second half is a formal way to provide a (counter)example. Usually you don't need to be this formal (could just state that a is in B but not A and instead of the second half of the proof).

Power Sets

The power set of A is denoted $\mathcal{P}(A)$ or 2^A

$$\mathcal{P}(A) = 2^A = \{B \mid B \subseteq A\}$$

The power set is a set of sets, where the elements are all of the possible subsets of A (including the empty set and A itself). I prefer the 2^A notation because it helps me remember how many elements there are supposed to be.

Example: Consider the set $X = \{a, b, c\}$. Then the power set of X is:

$$2^X = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\Big\}$$

You can think of power sets as potential menus. Perhaps X is the set of desserts that a restaurant can have, where $a =$ apple pie, $b =$ brownies, and $c =$ cheesecake. A restaurant might offer a menu with all three options, $\{a, b, c\}$. They might offer only brownies, $\{b\}$. They might also offer no dessert, $\emptyset$, which is cruel.

The power set represents, given a set of desserts, all of the possible menus that a restaurant might have. (Then we can start to think about the choice function – which dessert I'll pick as a function of the menu I get).

Note that if X has 3 elements, then the power set 2^X has $2^3 = 8$ elements.

Also note that every element of 2^X is a set. So if we want to discuss the $\{a, b\}$ menu:

- $A = \{a, b\}$
- $A \in 2^X$
- $A \subseteq X$
- NOT $\{a, b\} \subseteq 2^X$

Example: Proof using hypothesis in the conclusion
Prove that $2^A \subseteq 2^B$ if and only if $A \subseteq B$.

This is an if and only if, so we need to prove both directions:

$$2^A \subseteq 2^B \implies A \subseteq B$$

$$A \subseteq B \implies 2^A \subseteq 2^B$$

- Definition of subset: $A \subseteq B \iff (x \in A \Rightarrow x \in B)$
- Definition of power set: $2^A = \{C \mid C \subseteq A\}$
- Hypothesis in the conclusion: $P \Rightarrow (Q \Rightarrow R)$ iff $(P \wedge Q) \Rightarrow R$

To show ($\Rightarrow$): $A \subseteq B$

Proof:

Let $2^A \subseteq 2^B$ (by hypothesis)

- Definition of subset: $A \subseteq B \iff (x \in A \Rightarrow x \in B)$
- Definition of power set: $2^A = \{C \mid C \subseteq A\}$
- Hypothesis in the conclusion: $P \Rightarrow (Q \Rightarrow R)$ iff $(P \wedge Q) \Rightarrow R$

The other direction is a little more difficult.

$$A \subseteq B \implies 2^A \subseteq 2^B$$

Note that it can be rewritten :

$$A \subseteq B \implies \left[X \in 2^A \implies X \in 2^B\right]$$

This is our "hypothesis in the conclusion" logical form and is equivalent to:

$$\left[(A \subseteq B) \wedge (X \in 2^A)\right] \implies X \in 2^B$$

<u>To show ($\Leftarrow$):</u> $X \in 2^B$

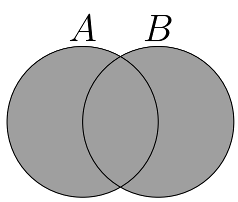
<u>Proof:</u>

Let $A \subseteq B$	(by hypothesis)
Let $X \in 2^A$	(by hypothesis)

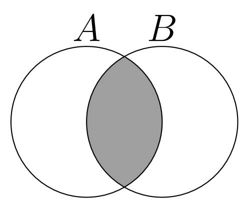
SET OPERATIONS

Consider sets A and B in universe U (so $A, B \subseteq U$):

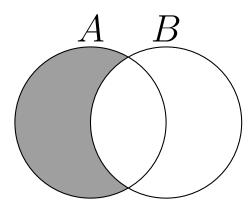
Union of A and B	$A \cup B$	$= \{x \mid x \in A \vee x \in B\}$
Intersection of A and B	$A \cap B$	$= \{x \mid x \in A \wedge x \in B\}$
Difference of A and B	$A - B$	$= \{x \mid x \in A \wedge x \notin B\}$ (also $A \backslash B$)
A and B are disjoint	iff	$A \cap B = \emptyset$
Complement of A	A^c	$A^c = U - A$



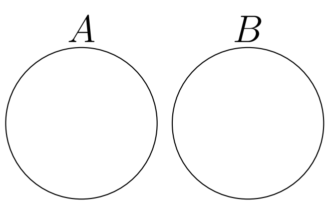
Union: $A \cup B$



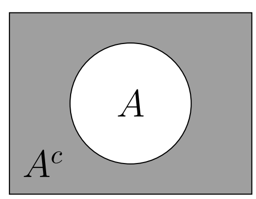
Intersection: $A \cap B$



Difference: $A - B$



Disjoint Sets



A and A^c

Example: Consider the sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$.
Let $U = \{1, 2, 3, 4, 5\}$. Then

- $A \cup B =$
- $A \cap B =$
- $A - B =$
- $B - A =$
- $A^c =$

Example: Proof by Contradiction
Let $A \cap C \subseteq B$ and $a \in C$. Prove that $a \notin A - B$.

- $A \subseteq B \iff \forall\, x, (x \in A \Rightarrow x \in B)$
- $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- $A - B = \{x \mid x \in A \wedge x \notin B\}$

To Show:

Proof:

Cardinality: The Basics

A **finite set** is one that has a finite number of elements. If you started labeling them from the 1^{st} element, to the 2^{nd} element, etc., then eventually you would get to the last element with a label $k \in \mathbb{N}$.

- $\{1, 2, 3, 4\}$
- $\{1, 2, 3, \ldots, 10^{100}\}$

(A finite set is always countable.)

A **countably infinite set** has an infinite number of elements and is countable. Countable means that we could map the elements to elements of $\mathbb{N}$. For a countably infinite set, we can still label the elements the 1^{st}, 2^{nd}, etc., though we would never be able to stop.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ is a countably infinite set.

$\mathbb{Z} = \{\ldots,$	-3,	-2,	-1,	0,	1,	2,	3,	$\ldots\}$	is also countably infinite.
etc.	6^{th}	4^{th}	2^{nd}	1^{st}	3^{rd}	5^{th}	7^{th}	etc.	

If you have a countably infinite set X, you can make it easier to work with by "labeling the elements" (map it to $\mathbb{N}$):

- $X = \{x_1, \ldots, x_n, \ldots\}$
- Define $f : X \to \mathbb{N}$ such that $f(x_n) = n$
- Now you can just think of X as though it were $\mathbb{N}$

A **uncountably infinite set** has infinite elements and is not countable.

- $[0, 1]$
- $(0, 1)$

You could make the first element 0, and the second element 0.1. But what about 0.01 and 0.001 and 0.0001? You see the problem.

Countability makes a big difference in utility theory. When the potential choices are countable, then we only have to consider how an element compares to its neighbors (make pairwise comparisons).

For example, if the universe of desserts is $\{apple\ pie,\ brownies,\ cheesecake, \ldots\}$, we just need to know how a compares to b and how b compares to c, etc. With transitivity, then we know what I'll choose from any menu.

When the potential choices are uncountable, we can't make a comparison between an element and its "neighbor" because there's always a closer neighbor. (What's the neighbor of 0?). In this case, we're going to want the additional property of continuity, which ensures that preferences are smooth across "neighborhoods."

Ordered Pairs & n-Tuples

So far we've thought about sets where the elements are scalars or single items.

- $A = \{1, 2, 3\}$
- $X = \{apple\ pie,\ brownies,\ cheesecake\}$
- $Y = \{espresso,\ milk,\ wine\}$
- $\mathbb{R}$

However, we can also have sets where the elements are ordered pairs. The version we are most familiar with is the ordered pair (x, y) from the two-dimensional Cartesian coordinate system $\mathbb{R}^2$.

Ordered pairs are elements in sets that are cross products.

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\} \qquad (a, b) \in A \times B$$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R} \wedge y \in \mathbb{R}\} \qquad (x, y) \in \mathbb{R}^2$$

$X \times Y$	$X \times X$
Useful for bundles	Useful for pairwise comparisons

Ordered pairs can be extended to n-tuples – for example, coordinates from the space $\mathbb{R}^n$.

More Proofs (Cases & Tautologies)

Recall that you can ...

- Separate $\vee$ into separate cases If $P\ldots$ and If $Q\ldots$
  (sometimes useful to also do: If $P \wedge Q\ldots$)

- $A \subseteq B \iff \forall\, x, (x \in A \Rightarrow x \in B)$
- $A \cup B = \{x \mid x \in A \vee x \in B\}$

Example: Let $A \subseteq C$ and $B \subseteq C$. Show that $A \cup B \subseteq C$

<u>To Prove:</u>
<u>Proof:</u>

Recall that you can ...

- Declare a tautology at any time $P \vee \neg P$
- Separate $\vee$ into separate cases If $P \ldots$ and If $Q \ldots$
  (sometimes useful to also do: If $P \wedge Q \ldots$)

Example: Let $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Show that $A \subseteq B$.

- $A \subseteq B \iff \forall\, x, (x \in A \Rightarrow x \in B)$
- $A \cup B = \{x \mid x \in A \vee x \in B\}$
- $A \cap B = \{x \mid x \in A \wedge x \in B\}$

To Show:
Proof:

Set Equivalences

$A \cup B$	$B \cup A$	(Commutative Laws)
$A \cap B$	$B \cap A$	”
$A \cup (B \cup C)$	$(A \cup B) \cup C$	(Associative Laws)
$A \cap (B \cap C)$	$(A \cap B) \cap C$	”
$A \cap (B \cup C)$	$(A \cap B) \cup (A \cap C)$	(Distributive Laws)
$A \cup (B \cap C)$	$(A \cup B) \cap (A \cup C)$	”
$(A \cup B)^c$	$A^c \cap B^c$	(DeMorgan’s Laws)
$(A \cap B)^c$	$A^c \cup B^c$	”
$(A^c)^c$	A	(logic)
$A \cup A^c$	U	(logic)
$A \cap A^c$	$\emptyset$	(logic)

Proof Strategies for Sets

$A \subseteq B$

- Let $x \in A$ (by hypothesis) – remember x is arbitrary
- Show that $x \in B$

$A = B$

- Show that $A \subseteq B$ and that $B \subseteq A$

$A \subset B$

- Show that $A \subseteq B$
- Find any $y \in B$ such that $y \notin A$

$A \cap B = \emptyset$

- Suppose that $x \in A \cap B$ (towards contradiction) – arbitrary x
- Show a contradiction

Moving Between Sets & Elements

$A \subseteq B$	$\forall x, (x \in A \Rightarrow x \in B)$
$A \cup B$	$\forall x, [(x \in A \cup B) \Rightarrow (x \in A \vee x \in B)]$
$A \cap B$	$\forall x, [(x \in A \cap B) \Rightarrow (x \in A \wedge x \in B)]$
$A - B$	$\forall x, [(x \in A - B) \Rightarrow (x \in A \wedge x \notin B)]$
$A \neq \emptyset$	$\exists\, x \in A$