WLOG: without loss of generality

let $x, y \in \mathbb{R}$

$\Rightarrow x \geq y \lor y \geq x$

Case 1: $x \geq y$

$\vdots$

Case 2: $y \geq x$

(similar to case 1)

let $x, y \in \mathbb{R}$

$\Rightarrow x \geq y \lor y \geq x$

WLOG, let $x \geq y$

$\vdots$

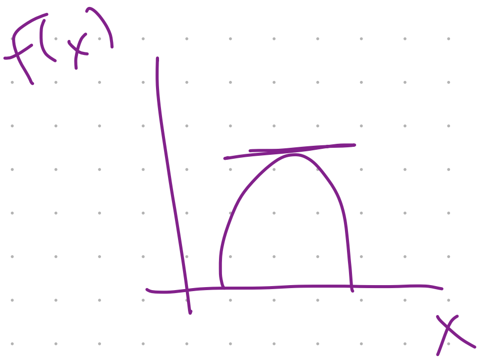
* make sure no generality is lost!

Last time

local max/min $\quad \frac{\partial f}{\partial x_i} = 0 \ \forall i$

↳ use FOC to find candidates

↳ use SOC / cunning & skill to determine min, max, saddle point



locally concave

$\approx$

local max

global min/max

↳ easy when function is concave/convex everywhere

$$\max_{x \in D(\theta)} f(x, \theta)$$

$f(\cdot)$ objective function

x choice variable

D choice set

θ parameter

$$x^*(\theta) = \arg\max_{x \in D(\theta)} f(x, \theta)$$

solution (set)

$V(\theta) = f(x^*, \theta)$ is the value function

Utility Maximization

$$\max_{x,y\geq 0} u(x,y) \quad \text{s.t.} \quad P_x \cdot x + P_y \cdot y = m$$

s.t. = subject to

pick x^*, y^* to maximize utility given your budget

$u(\cdot)$ utility function

- more is better (for now)

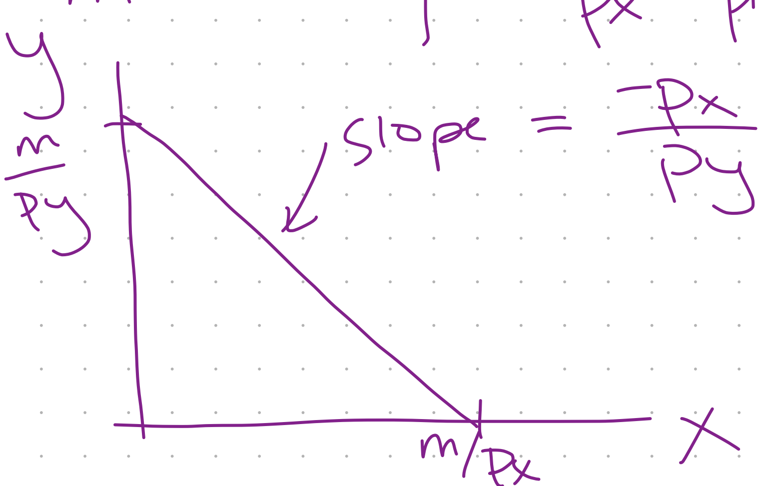
strong monotonicity assumption

$MU_x > 0 \qquad MU_y > 0$ (2 goods)

choice set: $x, y \in \mathbb{R}^+$ (no negative goods)

budget constraint

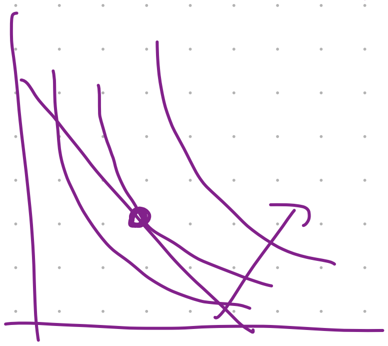
m: money $\quad P_x$: price of $x \quad P_y$: price of y



Strictly decreasing MRS (Cobb-Douglas)

$u(x,y) = x^{\alpha} y^{\beta}$

strictly convex IDCs
never cross axes,
not worried about
$x < 0, y < 0$



unique optimum
IDC is tangent to BC

MRS: −slope of IDC
how much y to give
up 1 x

−slope of BC: $\frac{P_x}{P_y}$ ← how much y I can buy
if I give up 1x

Reduces into 2 conditions

1. $MRS(x^*, y^*) = \frac{P_x}{P_y}$
2. $P_x \cdot x + P_y \cdot y = m$

a) set $MRS = \frac{P_x}{P_y}$
find $y = f(x)$

b) substitute $y = f(x)$
into the BC
solve for x^*

c) use $y = f(x)$
to find y^*

$u(x, y) = x \cdot y$

a) set $MRS = \frac{P_x}{P_y}$ & find $y = f(x)$

$$MRS = \frac{y}{x} = \frac{P_x}{P_y}$$

$$y = \frac{P_x \cdot x}{P_y}$$

b) substitute into BC

$$P_x \cdot x + P_y \cdot y = m$$

$$P_x \cdot x + P_y\left(\frac{P_x \cdot x}{P_y}\right) = m$$

$$P_x \cdot x + P_x \cdot x = m$$

$$\boxed{x^* = \frac{m}{2P_x}}$$

$$y^* = \frac{P_x \cdot x^*}{P_y} = \frac{P_x \cdot m}{P_y \cdot 2P_x} \boxed{= \frac{m}{2P_y}}$$

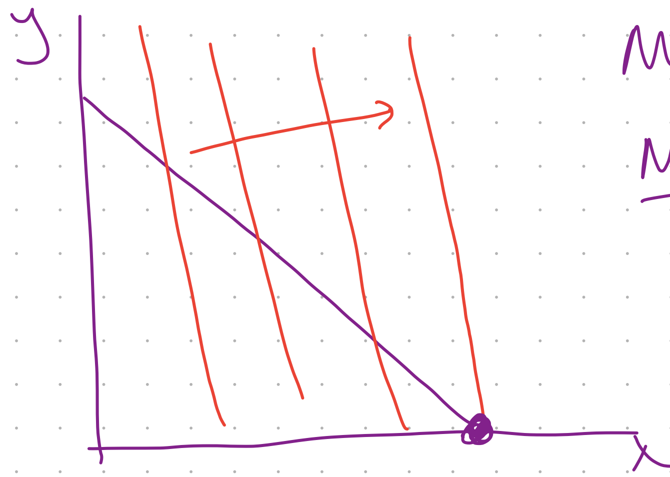
$u = x^\alpha y^\beta$

$$x^* = \frac{\alpha}{\alpha+\beta} \cdot \frac{m}{P_x} \qquad y^* = \frac{\beta}{\alpha+\beta} \cdot \frac{m}{P_y}$$

Constant MRS perfect substitutes

$u(x,y) = ax + by$ $\quad MRS = \frac{a}{b}$

Case 1:



$x^* = \frac{m}{P_x}$

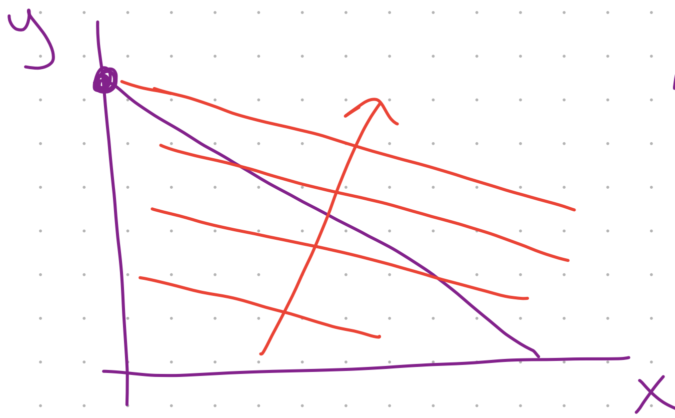
$y^* = 0$

$MRS > \frac{P_x}{P_y}$

$\frac{MU_x}{P_x} > \frac{MU_y}{P_y}$

bang for buck method

Case 2:



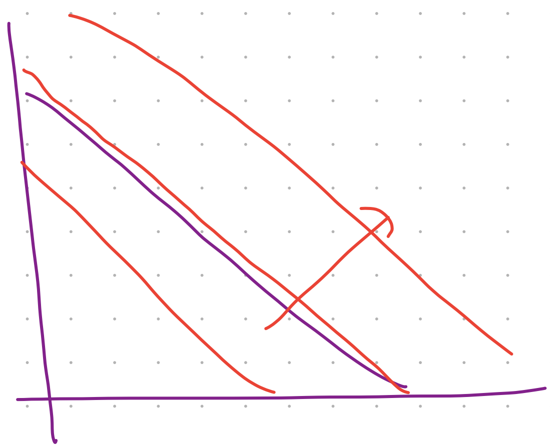
$x^* = 0$

$y^* = \frac{m}{P_y}$

$MRS < \frac{P_x}{P_y}$

$\frac{MU_x}{P_x} < \frac{MU_y}{P_y}$

Case 3:



$MRS = \frac{P_x}{P_y}$

$\underbrace{(x^*, y^*)}_{\text{choice correspondence}} = \{(x,y) \mid P_x \cdot x + P_y \cdot y = m\}$

$u(x,y) = 3x + y$

$MRS = 3$

What's the optimum if

$P_x = 4, \quad P_y = 3, \quad m = 12?$

$$MRS = 3 > \frac{4}{3} = \frac{P_x}{P_y}$$

$$x^* = \frac{m}{P_x} = \frac{12}{4} = 3 \qquad y^* = 0$$

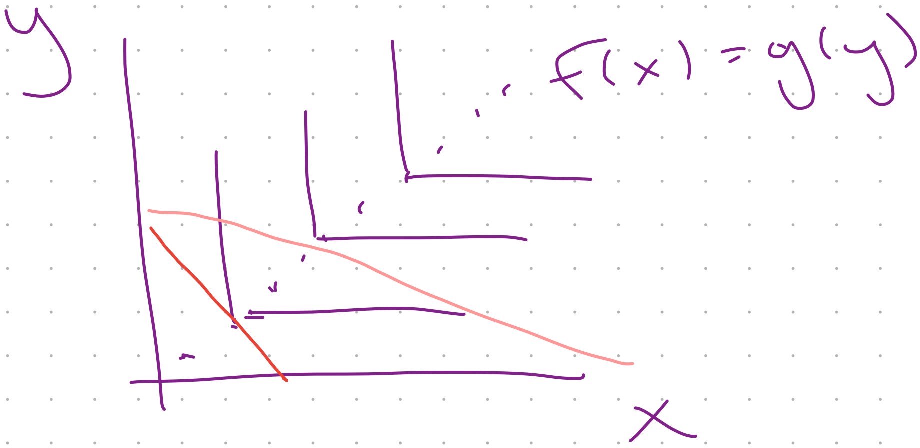
$P_x = 4 \quad P_y = 1 \quad m = 12?$

$$\frac{MU_x}{P_x} = \frac{3}{4} < \frac{MU_y}{P_y} = 1$$

$$x^* = 0 \qquad y^* = \frac{m}{P_y} = 12$$

Perfect complements (min function)

$u = \min\{f(x), g(y)\}$



2 conditions

① $f(x) = g(y)$ no wasted stuff

② $p_x \cdot x + p_y \cdot y = m$

$u(x,y) = \min\{2x, y\}$ y: tires x: bike frames

$2x = y$

$p_x \cdot x + p_y \cdot y = m$

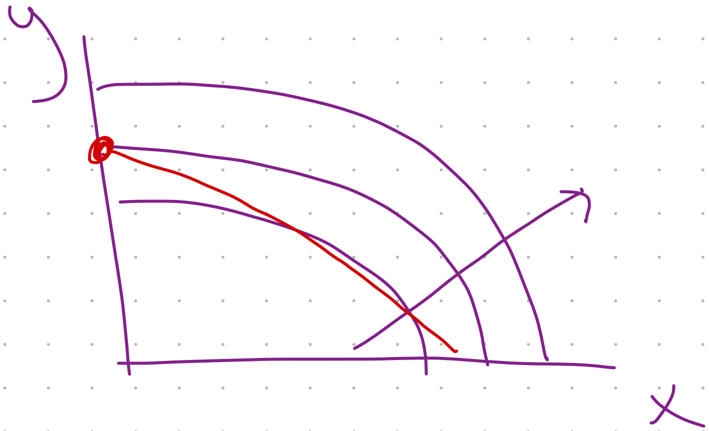
$p_x \cdot x + p_y \cdot 2x = m$

$x(p_x + 2p_y) = m$

$\boxed{x^* = \dfrac{m}{p_x + 2p_y}}$

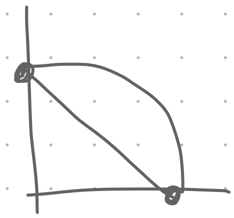
$\boxed{y^* = \dfrac{2m}{p_x + 2p_y}}$

Prefers Extremes (increasing MRS)



IDCs are concave shape

quasiconvex $u(\cdot)$



NOT tangency condition

Could have $x^* = \{(\frac{m}{p_x}, 0), (0, \frac{m}{p_y})\}$

$u(x,y) = x^2 + y^2$

Solution is a corner

either $(\frac{m}{p_x}, 0)$ or $(0, \frac{m}{p_y})$

whichever has greater utility

$p_x = 1 \quad p_y = 2 \quad m = 10$

$u(\frac{m}{p_x}, 0) = u(\frac{10}{1}, 0) = 10^2 = 100$

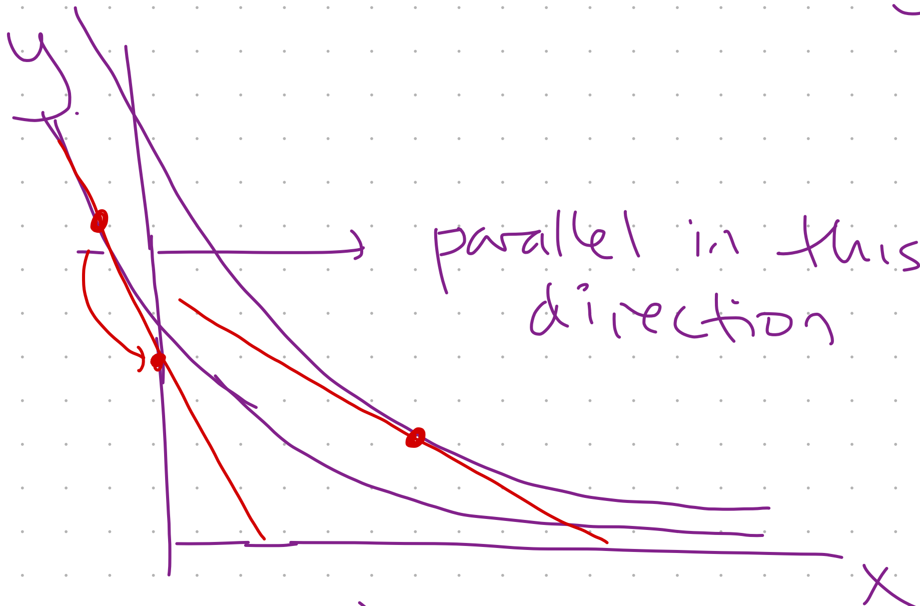
$u(0, \frac{m}{p_y}) = u(0, \frac{10}{2}) = 5^2 = 25$

$\boxed{(\frac{m}{p_x}, 0)} = \boxed{(10, 0)}$

Diminishing but not strictly MRS (quasilinear)

$u(x,y) = x + \ln(y)$

concave function



solve like C-D
if $x<0$ or $y<0$, move to nearest corner

① use $MRS = \frac{p_x}{p_y}$

② use BC

$u = x + \ln(y)$

$MRS = \frac{1}{1/y} = y = \frac{p_x}{p_y}$

$p_x \cdot x + p_y \cdot y = m$

$p_x \cdot x + p_y \cdot \frac{p_x}{p_y} = m$

$x = \frac{m - p_x}{p_x} \qquad y = \frac{p_x}{p_y}$

If $p_x = 20$, $p_y = 10$, $m = 50$?

$x^* = \frac{50-20}{20} = 1.5 \qquad y^* = \frac{20}{10} = 2$ ✓

If $p_x = 20$, $p_y = 10$, $m = 10$?

$x = \frac{10-20}{20} < 0$

↳ $x^* = 0 \qquad y^* = \frac{m}{p_y} = \frac{10}{10} = 1$

$$u = x + \ln(y)$$

$$x^* = \begin{cases} \dfrac{m - p_x}{p_x} & \text{if } m \geq p_x \ * \\ 0 & \text{otherwise} \end{cases}$$

$$y^* = \begin{cases} \dfrac{p_x}{p_y} & \text{if } m \geq p_x \ * \\ \dfrac{m}{p_x} & \text{otherwise} \end{cases}$$

$m \geq p_x$: I can afford the y^* from $MRS = \dfrac{p_x}{p_y}$

from a low m, you buy all y at first (high MU relative to x)

eventually, you achieve the optimal y

Then you spend the rest of your money on x

x is cash

Strong monotonicity. more is better

$MU_x > 0 \quad MU_y > 0$

both are goods

---

what if only one is a good?

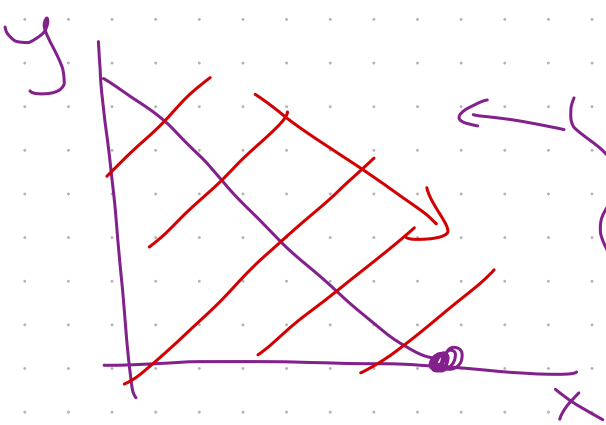
$u(x,y) = \ln(x) \quad MU_x = \frac{1}{x} > 0$

$MU_y = 0$ ← neither good nor bad

$u(x,y) = \ln(x) - y^2 \quad MU_y = -2y < 0$

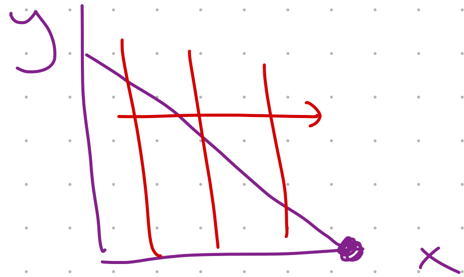
bad

You're going to spend all your money on the good $\left(\frac{m}{P_x}, 0\right)$



$u = x - y$

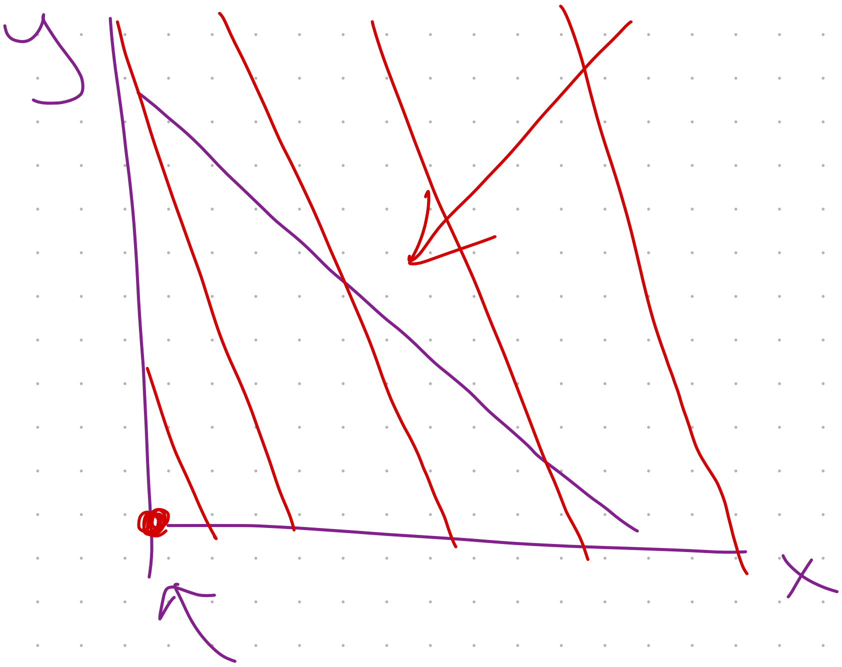
$MRS = -1$



$u = x$ →

$u = -x - y$

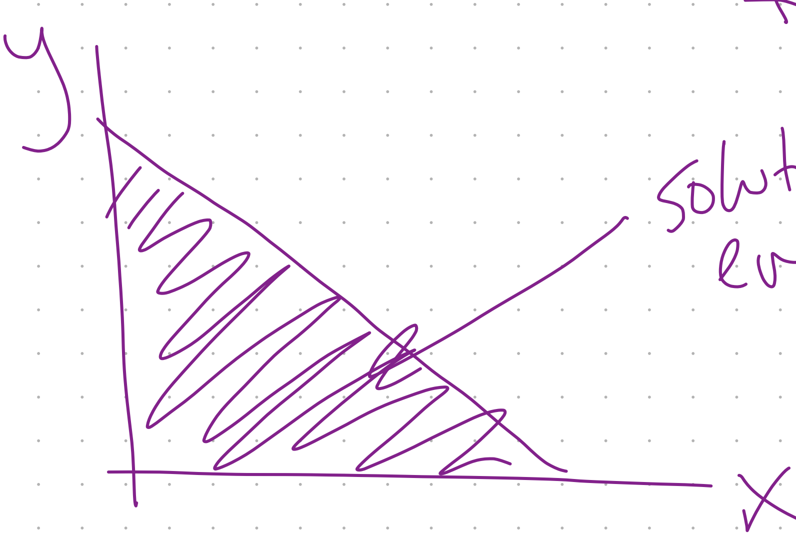
both are bads



$MRS = 1$

both are bads, then optimum is $(0,0)$

$u = 3$ neither good nor bad for either



solution set is everywhere in the budget set