Example: Use Kuhn-Tucker to solve, where $p_x, p_y, m > 0$:

Cobb-Douglas

$x = \frac{m}{2p_x}$ $y = \frac{m}{2p_y}$

$$\max_{x,y} \; xy \quad s.t. \quad m \geq p_x x + p_y y \quad x \geq 0 \quad y \geq 0$$

Lagrangian: $\mathcal{L} = xy + \lambda[m - p_x \cdot x - p_y y] + \mu_x(x-0) + \mu_y(y-0)$

(i) FOCs:

$\mathcal{L}_x = y - \lambda p_x + \mu_x = 0$

$\mathcal{L}_y = x - \lambda p_y + \mu_y = 0$

(ii) Constraints:

$m \geq p_x x + p_y \cdot y \qquad x \geq 0 \qquad y \geq 0$

(iii) Complementary slackness conditions:

$\lambda[m - p_x \cdot x - p_y \cdot y] = 0 \qquad \mu_x(x-0) = 0 \qquad \mu_y(y-0) = 0$

(iv) Non-negative multipliers:

$\lambda \geq 0 \qquad \mu_x \geq 0 \qquad \mu_y \geq 0$

1. No constraints bind
2. The budget constraint binds
3. $x \geq 0$ binds
4. $y \geq 0$ binds
5. Budget constraint and $x \geq 0$ bind
6. Budget constraint and $y \geq 0$ bind
7. $x \geq 0$ and $y \geq 0$ bind
8. All constraints bind

① no constraints bind

$m > p_x \cdot x + p_y \cdot y \qquad x > 0 \qquad y > 0$

$\lambda = 0 \qquad \mu_x = 0 \qquad \mu_y = 0$

$$\mathcal{L}_x = y - p_x \cdot \lambda + \mu_x = 0$$

$$y - 0 + 0 = 0$$

$$y = 0$$

contradiction!

② BC binds

$m = p_x x + p_y \cdot y \qquad x > 0 \qquad y > 0$

$\lambda > 0 \qquad \mu_x = 0 \qquad \mu_y = 0$

$$\mathcal{L}_x = y - \lambda p_x + \mu_x = 0$$

$$y - \lambda p_x = 0 \qquad \lambda = \frac{y}{p_x}$$

$$y = \frac{p_x \cdot x}{p_y}$$

$$\mathcal{L}_y = x - \lambda \cdot p_y + \mu_y = 0$$

$$x - \lambda p_y = 0 \qquad \lambda = \frac{x}{p_y}$$

$$m = p_x \cdot x + p_y \cdot \frac{p_x \cdot x}{p_y}$$

$$\boxed{x^* = \frac{m}{2p_x} \qquad y^* = \frac{m}{2p_y}}$$

③ $x \geq 0$ binds

$m > p_x x + p_y \cdot y \qquad x = 0 \qquad y > 0$

$\lambda = 0 \qquad \mu_x > 0 \qquad \mu_y = 0$

$$\mathcal{L}_x = y - \lambda \cdot p_x + \mu_x = 0$$

$$y + \mu_x = 0 \qquad y = -\mu_x$$

④ $y \geq 0$ binds

$m > p_x \cdot x + p_y \cdot y$ $\quad x > 0$ $\quad y = 0$

$\lambda = 0$ $\quad \mu_x = 0$ $\quad \mu_y > 0$

$\mathcal{L}_x = y - p_x \cdot \lambda + \mu_x = 0$

$y = 0$

$\mathcal{L}_y = x - \lambda \cdot p_y + \mu_y = 0$ $\quad$ X

$x + \mu_y = 0$ $\quad x = -\mu_y$

⑤ BC & $x \geq 0$ binds

$m = p_x \cdot x + p_y \cdot y$ $\quad x = 0$ $\quad y > 0$

$\lambda > 0$ $\quad \mu_x > 0$ $\quad \mu_y = 0$

$\mathcal{L}_x = y - p_x \cdot \lambda + \mu_x = 0$

$\mathcal{L}_y = x - \lambda \cdot p_y + \mu_y = 0$ $\quad x - \lambda p_y = 0$

$0 - \lambda p_y + 0 = 0$ $\quad$ X $\quad x = \lambda \cdot p_y$

$x > 0$

$-\lambda \cdot p_y = 0$ $\quad \lambda = 0$ $\quad x = 0$

⑥ BC & $y \geq 0$ binds

$m = p_x \cdot x + p_y \cdot y$ $\quad x > 0$ $\quad y = 0$

$\lambda > 0$ $\quad \mu_x = 0$ $\quad \mu_y > 0$

$\mathcal{L}_x = y - p_x \cdot \lambda + \mu_x = 0$ $\quad \lambda = 0$ $\quad$ X

$0 - \lambda p_x + 0 = 0$

7) $x \geq 0$ & $y \geq 0$ bind

$m > p_x \cdot x + p_y \cdot y \qquad x = 0 \qquad y = 0$

$\lambda = 0 \qquad \mu_x > 0 \qquad \mu_y > 0$

$$\mathcal{L}_x = y - p_x \cdot \lambda + \mu_x = 0$$

$$0 - 0 + \mu_x = 0$$

8) all constraints bind

$m = p_x \cdot x + p_y \cdot y \qquad x = 0 \qquad y = 0$

$$m = p_x \cdot 0 + p_y \cdot 0$$

$$m = 0$$

$$\boxed{x^* = \frac{m}{2p_x} \qquad y^* = \frac{m}{2p_y}}$$

maximum yay!

Example: Use Kuhn-Tucker to solve, where $p_x, p_y, m > 0$:

$$\max_{x,y} \; x + y \quad s.t. \quad m \geq p_x x + p_y y \quad x \geq 0 \quad y \geq 0$$

$$\mathcal{L} = x + y + \lambda[m - p_x \cdot x - p_y \cdot y] + \mu_x x + \mu_y \cdot y$$

$$\mathcal{L}_x = 1 - p_x \cdot \lambda + \mu_x = 0$$

$$\mathcal{L}_y = 1 - p_y \cdot \lambda + \mu_y = 0$$

1 no constraints bind

$m > p_x \cdot x + p_y \cdot y$ $\quad x > 0 \quad y > 0$

$\lambda = 0 \quad \mu_x = 0 \quad \mu_y = 0$

$\mathcal{L}_x = 1 - 0 + 0 = 0$ ✗

2 BC binds

$m = p_x \cdot x + p_y \cdot y$ $\quad x > 0 \quad y > 0$

$\lambda > 0 \quad \mu_x = 0 \quad \mu_y = 0$

$\mathcal{L}_x = 1 - p_x \cdot \lambda + \mu_x = 0$

$1 - p_x \cdot \lambda = 0 \quad p_x \cdot \lambda = 1 \quad \lambda = \frac{1}{p_x}$

$\mathcal{L}_y = 1 - p_y \cdot \lambda + \mu_y = 0 \quad \lambda = \frac{1}{p_y}$

$\boxed{p_x = p_y}$ $\quad m = p_x \cdot x + p_y \cdot y$ any $x, y \Rightarrow p_x x + p_y y = m$ is a solution

$$\mathcal{L}_x = 1 - p_x \cdot \lambda + \mu_x = 0$$

$$\mathcal{L}_y = 1 - p_y \cdot \lambda + \mu_y = 0$$

3 $\underline{x \geq 0 \quad \text{binds}}$

$m > p_x \cdot x + p_y \cdot y \qquad x = 0 \qquad y > 0$

$\lambda = 0 \qquad \mu_x > 0 \qquad \mu_y = 0$

$$\mathcal{L}_x = 1 - 0 + \mu_x = 0$$

$$1 = -\mu_x$$

4 $\underline{y \geq 0 \quad \text{binds}}$

$m > p_x \cdot x + p_y \cdot y \qquad x > 0 \qquad y = 0$

$\lambda = 0 \qquad \mu_x = 0 \qquad \mu_y > 0$

$$\mathcal{L}_x = 1 - 0 + 0 = 0$$

$$1 = 0$$

$\mathcal{L}_x = 1 - p_x \cdot \lambda + \mu_x = 0$

$\mathcal{L}_y = 1 - p_y \cdot \lambda + \mu_y = 0$

⑤ BC & $x \geq 0$ binds

$m = p_x \cdot x + p_y \cdot y$ $\quad x = 0 \quad y > 0$

$\lambda > 0 \quad \mu_x > 0 \quad \mu_y = 0$

$\mathcal{L}_y = 1 - p_y \cdot \lambda = 0 \quad p_y \lambda = 1 \quad \lambda = \frac{1}{p_y}$

$\mathcal{L}_x = 1 - p_x \lambda + \mu_x = 0$

$1 - \frac{p_x}{p_y} + \mu_x = 0$

$\mu_x = \frac{p_x}{p_y} - 1$ $\quad$ negative if $\frac{p_x}{p_y} \leq 1$

If $p_x > p_y$, then $x^* = 0$

$m = p_x \cdot x + p_y \cdot y$ $\quad y^* = \frac{m}{p_y}$

$\mathcal{L}_x = 1 - p_x \cdot \lambda + \mu_x = 0$

$\mathcal{L}_y = 1 - p_y \cdot \lambda + \mu_y = 0$

⑥ BC & $y \geq 0$ bind

$m = p_x \cdot x + p_y \cdot y \qquad x > 0 \qquad y = 0$

$\lambda > 0 \qquad \mu_x = 0 \qquad \mu_y > 0$

$\mathcal{L}_x = 1 - p_x \cdot \lambda = 0 \qquad \lambda = \frac{1}{p_x}$

$\mathcal{L}_y = 1 - p_y \cdot \lambda + \mu_y = 0$

$1 - \frac{p_y}{p_x} + \mu_y = 0$

$\mu_y = \frac{p_y}{p_x} - 1 > 0$

If $p_y > p_x \qquad y^* = 0 \qquad x^* = \frac{m}{p_x}$

$\mathcal{L}_x = 1 - p_x \cdot \lambda + \mu_x = 0$

$\mathcal{L}_y = 1 - p_y \cdot \lambda + \mu_y = 0$

⑦ $x \geq 0$ & $y \geq 0$ bind

$m > p_x \cdot x + p_y \cdot y$ $\quad x = 0 \quad y = 0$

$\lambda = 0$ $\quad \mu_x > 0 \quad \mu_y > 0$

$\mathcal{L}_x = 1 - 0 + \mu_x = 0$

$\mu_x = -1$ $\quad$ X

⑧ all constraints bind

$m = p_x \cdot x + p_y \cdot y$ $\quad x = 0 \quad y = 0$

$m = 0 + 0$ $\quad m = 0$ $\quad$ X