Required Problems

1. **Using truth tables, prove both of DeMorgan's Laws for logical connectives.**

   (a) $\neg(P \wedge Q)$ **is logically equivalent to** $\neg P \vee \neg Q$

P	Q	$\neg P$	$\neg Q$	$(P \wedge Q)$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

   The last two columns, for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ respectively, have the same truth values for all truth assignments of P and Q; thus, they are logically equivalent.

   (b) $\neg(P \vee Q)$ **is logically equivalent to** $\neg P \wedge \neg Q$

P	Q	$\neg P$	$\neg Q$	$(P \vee Q)$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

   Once again, the last two columns, for $\neg(P \vee Q)$ and $\neg P \wedge \neg Q$ respectively, have the same truth values for all truth assignments of P and Q; thus, they are logically equivalent.

2. **Let x and y be integers. Prove that if x and y are even, then $x + y$ is even.**

   - $\mathbb{Z}$ is the set of integers
   - Def. of even: $x \in \mathbb{Z}$ is even iff $\exists\, k \in \mathbb{Z} \ni x = 2k$

   To show: There exists an integer h such that $x + y = 2h$

   Proof:

   Let $x, y \in \mathbb{Z}$ be even (by hypothesis)

   $\implies \big(\exists\, k \in \mathbb{Z} \ni x = 2k\big) \wedge \big(\exists\, j \in \mathbb{Z} \ni y = 2j\big)$ (by def. of even)

   $\implies x + y = 2k + 2j$ (summing)

   $\implies x + y = 2(k + j)$ (by distributivity)

   $k + j \in \mathbb{Z}$ (by closure)

   Let $h = k + j$ (defining an integer h)

   $\implies \exists\, h \in \mathbb{Z} \ni x + y = 2h$ (substituting for $j + k$)

   $\implies x + y$ is even (by def. of even)

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3. **Rewrite each of the following sentences to be symbolic sentences using logical connectives and quantifiers. If a quantifier's universe is included in the English sentence, be sure to include it in the symbolic sentence.**

   (a) **If $x = 1$ or $x = -1$, then $|x| = 1$.**

$$\big(x = 1 \vee x = -1\big) \implies |x| = 1$$

(b) **B is invertible is a necessary and sufficient condition for $|B| \neq 0$.**

$$\exists\, B^{-1} \iff |B| \neq 0$$

(c) **$6 \geq n-3$ only if $n > 8$ or $n = 9$.**

$$\neg\big(n > 8 \lor n = 9\big) \implies \neg\big(6 \geq n-3\big)$$

(d) **Every nonzero real number is positive or negative.**

$$\forall\, x \in \mathbb{R} \ni x \neq 0, (x > 0 \lor x < 0)$$

(e) **S is compact iff S is closed and bounded.**

$$(S \text{ is compact}) \iff (S \text{ is closed and bounded})$$

(f) **Every integer is greater than some integer.**

$$\forall\, x \in \mathbb{Z},\ \exists\, y \in \mathbb{Z} \ni x > y$$

4. **Let A and B be sets. Prove that $A \subseteq B$ if and only if $A - B = \emptyset$.**

- Statement using biconditional: $(A \subseteq B) \iff (A - B = \emptyset)$
- Def. of subset: $A \subseteq B \iff \big(x \in A \implies x \in B\big)$
- Def. of set difference: $A - B = \{x | x \in A \land x \notin B\}$
- Theorem (T1): $(P \implies Q)$ is logically equivalent to $(\neg P) \lor Q$
- Theorem (T2): For all sets A, $\emptyset \subseteq A$
- Contrapositive of ($\Rightarrow$): $A - B \neq \emptyset \implies A \not\subseteq B$

Proof by contraposition, to show: $A \not\subseteq B$.
Proof ($\Rightarrow$):

$$\begin{aligned}
&\text{Let } A - B \neq \emptyset && \text{(by hypothesis)}\\
&\implies \exists\, x \in A - B && \text{(by def. of non-empty)}\\
&\implies x \in A \land x \notin B && \text{(by def. of set difference)}\\
&\implies \neg\big(x \notin A \lor x \in B\big) && \text{(by negation)}\\
&\implies \neg\big(x \in A \implies \in B\big) && \text{(by T1)}\\
&\implies \neg(A \subseteq B) && \text{(by def. of subset)}\\
&\implies A \not\subseteq B && \text{(by negation)}
\end{aligned}$$

To show: $A \subseteq B$.
Proof ($\Leftarrow$):

$$
\begin{aligned}
&\text{Let } A - B = \emptyset && \text{(by hypothesis)}\\
&\text{Case 1: } A = \emptyset && \text{(by hypothesis)}\\
&\implies A \subseteq B && \text{(by T2)}\\
&\text{Case 2: } A \neq \emptyset && \text{(by hypothesis)}\\
&\implies \exists\, x \in A && \text{(by def. of non-empty)}\\
&\implies x \notin A - B && \text{(by def. of empty)}\\
&\implies \neg(x \in A \wedge x \notin B) && \text{(by def. of set difference)}\\
&\implies x \notin A \vee x \in B && \text{(by negation)}\\
&\implies (x \in A \implies x \in B) && \text{(by T1)}\\
&\implies A \subseteq B && \text{(by def. of subset)}
\end{aligned}
$$

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5. **In class, I defined the uniqueness existential quantifier so that $\exists!$ means "there exists a unique". However, it can actually be defined using the symbols we already had, $\wedge, \forall, \exists$, etc. Write a symbolic sentence that is equivalent to $\exists! x \ni A(x)$ without using !.**

   The quantifier $\exists!$ states that there exists an element making $A(x)$ true, and if $A(x)$ is true for two values, those values must be the same. In other words,

$$\exists!\, x \ni A(x) \iff \big(\exists x \ni A(x)\big) \wedge \big(\forall y \wedge \forall z, A(y) \wedge A(z) \implies y = z\big)$$

6. **Let $A = \{1,3,5,7,9\}$, $B = \{0,2,4,6,8\}$, and $C = \{1,2,4,5,7,8\}$ and $D = \{1,2,3,5,6,7,8,9,10\}$. Find the following:**

   (a) $A \cup B$

$$A \cup B = \{0,1,2,3,4,5,6,7,8,9\}$$

   (b) $A - B$

$$A - B = \{1,3,5,7,9\}$$

   (c) $(A \cap C) \cap D$

$$(A \cap C) \cap D = \{1,5,7\}$$

   (d) $A \cup (C \cap D)$

$$A \cup (C \cap D) = \{1,2,3,5,7,8,9\}$$

Optional Problems

7. **Let x and y be integers. Prove the following propositions:**

   (a) **If x and y are even, then xy is even.**

   To show: there exists an integer h such that $xy = 2h$.
   Proof:

Let x and y be even integers (by hypothesis)
$\implies \Big(\exists k \in \mathbb{Z} \ni x = 2k\Big) \wedge \Big(\exists j \in \mathbb{Z} \ni y = 2j\Big)$ (by def. of even)
$\implies xy = (2j)(2k)$ (multiplying)
$\implies xy = 2(2jk)$ (by associativity)
$2jk \in \mathbb{Z}$ (by closure)
Let $h = 2jk$ (defining an integer h)
$\implies xy = 2h$ (substituting for $2jk$)
$\implies xy$ is even (by def. of even)

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(b) **If x and y are odd, then $x + y$ is even.**

To show: there exists an integer h such that $x + y = 2h$
Proof:

Let x and y be odd integers (by hypothesis)
$\implies \Big(\exists k \in \mathbb{Z} \ni x = 2k + 1\Big) \wedge \Big(\exists j \in \mathbb{Z} \ni y = 2j + 1\Big)$ (by def. of odd)
$\implies x + y = (2k + 1) + (2j + 1)$ (summing)
$\implies x + y = 2k + 2j + 2$ (by associativity/commutativity)
$\implies x + y = 2(k + j + 1)$ (by distributivity)
$k + j + 1 \in \mathbb{Z}$ (by closure)
Let $h = k + j + 1$ (defining an integer h)
$\implies x + y = 2h$ (substituting for $k + j + 1$)
$\implies x + y$ is even (by def. of even)

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(c) **If x is even and y is odd, then $x + y$ is odd.**

To show: there exists an integer h such that $x + y = 2h + 1$
Proof:

Let x be an even integer and let y be an odd integer (by hypothesis)
$\implies \Big(\exists k \in \mathbb{Z} \ni x = 2k\Big) \wedge \Big(\exists j \in \mathbb{Z} \ni y = 2j + 1\Big)$ (by def. of even and odd)
$\implies x + y = 2k + 2j + 1$ (summing)
$\implies x + y = 2(k + j) + 1$ (by distributivity)
$k + j \in \mathbb{Z}$ (by closure)
Let $h = k + j$ (defining an integer h)
$\implies x + y = 2h + 1$ (substituting for $k + j$)
$\implies x + y$ is odd (by def. of odd)

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8. **Let x be an integer. Write a proof by contraposition to show that if x is even, then $x+1$ is odd.**

   - Contrapositive: if $x+1$ is not odd, then x is not even
   - Theorem (T1): x is not even if and only if x is odd.

   Proof by contraposition to show: x is not even
   Proof:

Let x be an integer such that $x+1$ is not odd	(by hypothesis)
$\implies x+1$ is even	(by T1)
$\implies \exists\, k \in \mathbb{Z} \ni x+1 = 2k$	(by def. of even)
$\implies x = 2k-1$	(subtracting 1 from both sides)
$\implies x = 2k-2+1$	(rearranging the r.h.s)
$\implies x = 2(k-1)+1$	(by distributivity)
$k-1$ is an integer	(by closure)
let $j = k-1$	(defining an integer j)
$\implies x = 2j+1$	(subsituting for $k-1$)
$\implies x$ is odd	(by def. of odd)
$\implies x$ is not even	(by T1)

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9. **Suppose a and b are positive integers. Write a proof by contradiction to show that if ab is odd, the both a and b are odd.**

   - Theorem (T1): x is not even if and only if x is odd.

   Proof by contradiction to show: if a or b is not odd, then ab is odd and not odd.
   Proof:

Let $ab \in \mathbb{Z}_+ \ni ab$ is odd and a or b is not odd	(towards a contradiction)
$\implies a$ is even or b is even	(by T1)
Case 1: a is even	
$\implies \exists\, k \in \mathbb{Z} \ni a = 2k$	(by def. of even)
$\implies ab = (2k)b$	(substituting for a)
$\implies ab = 2(kb)$	(by associativity)
kb is an integer	(by closure)
$\implies ab$ is even	(by def. of even)
$\implies ab$is not odd	
Thus, a contradiction	
Case 2: b is even	
(similar to Case 1)	
Case 3: both a and b are even	
$\implies ab$ is even	(by proof in problem 10(a))
$\implies ab$ is not odd	(by T1)
Thus, a contradiction	

   Because every case produces the contradiction ab is odd and not odd, it must be the case that a and b are both odd. ■

10. **Let A, B, C, and D be sets prove that if $C \subseteq A$ and $D \subseteq B$ and A and B are disjoint, then C and D are disjoint.**

- Def. of disjoint: A and B are disjoint $\iff A \cap B = \emptyset$
- Contrapositive: C and D are not disjoint implies A and B are not disjoint

Proof by contraposition to show: A and B are not disjoint
Proof:

Let $C \subseteq A$ and $D \subseteq B$ such that C and D are not disjoint (by hypothesis)
$\implies C \cap D \neq \emptyset$ (by def. of disjoint)
$\implies \exists\, x \in C \cap D$ (by def. of non-empty)
$\implies x \in C \wedge x \in D$ (by def. of $\cap$)
$\implies x \in A \wedge x \in B$ (by def. of subset)
$\implies x \in A \cap B$ (by def. of $\cap$)
$\implies A \cap B \neq \emptyset$ (by def. of non-empty)
$\implies A$ and B are not disjoint (by def. of disjoint)

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11. **Find the contrapositive and converse of each of the following statements:**

(a) **"If squares have four sides, then triangles have four sides."**

This has the form: (squares have four sides) $\implies$ (trianges have four sides).

- Contrapositive: "If triangles do not have four sides, then squares do not have four sides."
- Converse: "If triangles have four sides, then squares have four sides."

(b) **"A sequence a is bounded whenever a is convergent."**

This has the form: (a is convergent) $\implies$ (a is bounded).

- Contrapositive: "If a is not bounded, then a is not convergent."
- Converse: "If a is bounded, then a is convergent."

(c) **"The differentiability of a function f is sufficient for f to be continuous.**

This has the form: (f is differentiable) $\implies$ (f is continuous)

- Contrapositive: "If f is not continuous, then f is not differentiable."
- Converse: "If f is continuous, then f is differentiable."