Required Problems

1. **Consider the sequence $\{x_n\}_{n=1}^{\infty}$ such that**

$$x_n = \frac{n+1}{n}$$

To what does this sequence converge? Prove that this sequence converges to that limit.

This sequence converges to 1, i.e., $x_n \to 1$.
To show: $|\frac{n+1}{n} - 1| < \varepsilon$.
Proof:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 && \text{(by hypothesis)}\\
&\text{Let } N > \frac{1}{\varepsilon} \text{ and } n > N && \text{(by hypothesis)}\\
&\left|\frac{n+1}{n} - 1\right| = \left|\frac{1}{n}\right| && \text{(simplifying)}\\
&= \frac{1}{n} && \text{(by } n > 0\text{)}\\
&< \frac{1}{N} && \text{(by } n > N\text{)}\\
&< \frac{1}{1/\varepsilon} && \text{(by } N > 1/\varepsilon\text{)}\\
&= \varepsilon && \text{(simplifying)}
\end{aligned}$$

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2. **Let S and T be convex sets. Prove that the intersection of S and T is also a convex set.**

To show: $t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in S \cap T$
Proof:

$$\begin{aligned}
&\text{Let } S \text{ and } T \text{ be convex sets, } \mathbf{x}_1, \mathbf{x}_2 \in S \cap T, \text{ and } t \in [0,1] && \text{(by hypothesis)}\\
&\implies \Big(\mathbf{x}_1 \in S \wedge \mathbf{x}_1 \in T\Big) \wedge \Big(\mathbf{x}_2 \in S \wedge \mathbf{x}_2 \in T\Big) && \text{(by def. of } \cap\text{)}\\
&\implies \Big(\mathbf{x}_1 \in S \wedge \mathbf{x}_2 \in S\Big) \wedge \Big(\mathbf{x}_1 \in T \wedge \mathbf{x}_2 \in T\Big) && \text{(by associativity)}\\
&\implies \Big(t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in S\Big) \wedge \Big(t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in T\Big) && \text{(by def. of convex)}\\
&\implies t\mathbf{x}_1 + (1-t)\mathbf{x}_2 \in S \cap T && \text{(by def. of } \cap\text{)}
\end{aligned}$$

■

3. **The set $S^{n-1} = \{\mathbf{x} \mid \sum_{i=1}^{n} x_i = 1 \;\wedge\; x_i \geq 0 \;\forall i = 1, \ldots, n\}$ is the $(n-1)$-dimensional unit simplex.**

(a) **Describe in words the set S^{n-1} for $n = 3$**

It is the set contained in a closed equilateral triangle with sides of length 1. The triangle connects points (1,0,0), (0,1,0), and (0,0,1) in $\mathbb{R}^3$.

(b) **Prove that S^{n-1} is a convex set.**

To show: $t\mathbf{x} + (1-t)\mathbf{y} \in S$
Proof:

Let $\mathbf{x}, \mathbf{y} \in S$ and $t \in [0,1]$	(by hypothesis)
Consider $t\mathbf{x} + (1-t)\mathbf{y}$	(the convex combo.)
$0 \leq tx_i + (1-t)y_i < 1 \quad \forall i = 1, \ldots, n$	(by $t \in [0,1]$)
$\sum_{i=1}^{n} \Big(tx_i + (1-t)y_i \Big)$	(summing the elements)
$= \sum_{i=1}^{n} tx_i + \sum_{i=1}^{n} (1-t)y_i$	(by associativity)
$= t \sum_{i=1}^{n} x_i + (1-t) \sum_{i=1}^{n} y_i$	(by distributivity)
$= t \cdot 1 + (1-t) \cdot 1$	(by $\mathbf{x}, \mathbf{y} \in S$)
$= 1$	(simplifying)
$\implies t\mathbf{x} + (1-t)\mathbf{y} \in S$	(by def. of S)

$\blacksquare$

(c) **Prove that S^{n-1} is a compact set.**

- Theorem (T1): $\mathbf{x}_k \to \mathbf{c} \iff x_{ik} \to c_i$ for all i (each element of the vector converges)
- Theorem (T2): $a_k \to a$ and $b_k \to b$ implies $a_k + b_k \to a + b$
- Theorem (T3): Weak inequalities are preserved in in the limit
- Lemma (L1): Constant sequences converge, i.e., $\{d, d, d \ldots\} \to d$

To show: $\mathbf{c} \in S$

Proof:

Let $\{\mathbf{x}_k\}_{k=0}^{\infty}$ be a sequence in $S \ni \big(\mathbf{x}_k \to \mathbf{c}\big) \wedge \big(\mathbf{x}_k \in S \; \forall \; k\big)$	(by hypothesis)
$\implies \Big(x_{ik} \to c_i, c_i \geq 0 \; \forall \; i\Big) \wedge \left(\sum_{i=1}^{n} x_{ik} = 1 \; \forall \; k\right) \wedge \Big(x_{ik} \geq 0 \; \forall \; i, k\Big)$	(by T1, T3, and $\mathbf{x}_k \in S$)
$\implies \left(\sum_{i=1}^{n} x_{ik} \to \sum_{i=1}^{n} c_i\right) \wedge \left(\sum_{i=1}^{n} x_{ik} \to 1\right)$	(by T2 and L1)
$\implies \sum_{i=1}^{n} c_i = 1$	(limits are unique)
$\implies \mathbf{c} \in S$	(by def. of S)

Because the convergent sequence and limit were arbitrary, it must be the case that the limit point of every convergent sequence is in S. Thus, S is closed.

To show: S is bounded

Proof:

Let $\mathbf{x} \in S$	(by hypothesis)
Let $M = 2$	(by hypothesis)
$\implies \sum_{i=1}^{n} x_i = 1 \wedge x_i \geq 0 \; \forall i$	(def of S)
$\implies 0 \leq x_i \leq 1 \; \forall i$	(algebra)
$\implies -M \leq x_i \leq M \; \forall i$	(algebra)
$\implies S$ is bounded	(def of bounded)

Thus, S is closed and bounded, implying it is compact. ■

4. **Let D be a convex subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$. For the following two statements, if it is true provide a proof. If it is false, provide a counterexample.**

(a) **f is strictly concave $\implies$ f is strictly quasiconcave**

To show: $f(t\mathbf{y} + (1-t)\mathbf{z}) > \min\{f(\mathbf{y}), f(\mathbf{z})\}$

Proof:

Let $\mathbf{y}, \mathbf{z} \in D \wedge t \in (0,1)$	(by hyp)
Consider $f(t\mathbf{y} + (1-t)\mathbf{z})$	(by hyp)
$> tf(\mathbf{y}) + (1-t)f(\mathbf{z})$	(def of strictly concave)
Case 1: $f(\mathbf{y}) \geq f(\mathbf{z})$	
$= t[f(\mathbf{y}) - f(\mathbf{z})] + f(\mathbf{z})$	(algebra)
$\geq f(\mathbf{z})$	$(f(\mathbf{y}) \geq f(\mathbf{z}))$
$= \min\{f(\mathbf{y}), f(\mathbf{z})\}$	$(f(\mathbf{y}) \geq f(\mathbf{z}))$
Case 2: $f(\mathbf{y}) \leq f(\mathbf{z})$	
Let $s = 1 - t$	(by hyp)
$= (1-s)f(\mathbf{y}) + sf(\mathbf{z})$	(def of s)
$= f(\mathbf{y}) + s[f(\mathbf{z}) - f(\mathbf{y})]$	(algebra)
$\geq f(\mathbf{y})$	$(f(\mathbf{y}) \leq f(\mathbf{z}))$
$= \min\{f(\mathbf{y}), f(\mathbf{z})\}$	$(f(\mathbf{y}) \leq f(\mathbf{z}))$
$\geq \min\{f(\mathbf{y}), f(\mathbf{z})\}$	(logic)

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(b) **f is strictly quasiconcave $\implies$ f is strictly concave**

False. Consider $f(x) = x$ where $f : \mathbb{R} \to \mathbb{R}$.

To show: $f(ty + (1-t)z) > \min(y, z)$

Proof:

Let $y, z \in \mathbb{R} \ni y \neq z \wedge t \in (0,1)$	(by hyp)
Consider $f(ty + (1-t)z)$	(by hyp)
$= ty + (1-t)z$	(def of f)
Case 1: $y > z$	
$= t(y-z) + z$	(algebra)
$> z$	$(t(y-z) > 0)$
$= \min(y, z)$	$(y > z)$
Case 2: $y < z$	
Let $s = 1 - t$	(by hyp)
$= (1-s)y + sz$	(def of s)
$= y + s(z - y)$	(algebra)
$> y$	$(s(z-y) > 0)$
$= \min(y, z)$	$(y < z)$
$\implies > \min(y, z)$	(logic)

But $f(x) = x$ is not strictly concave. Consider $y = 1$ and $z = 3$ and $t = 0.5$:

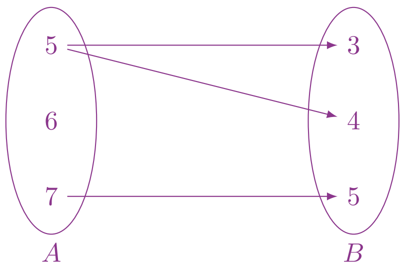
$$f(ty + (1-t)z) = f(0.5 + 1.5) = f(2) = 2$$

$$tf(y) + (1-t)f(y) = 0.5 * 1 + 0.5 * 3 = 2$$

Additional Practice Problems (I will provide solutions for these but not feedback)

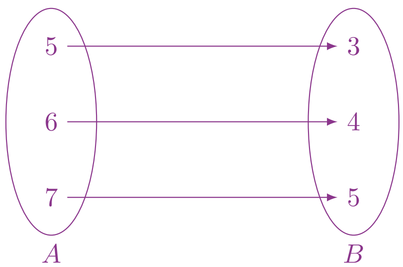
5. **Give a relation r from $A = \{5, 6, 7\}$ to $B = \{3, 4, 5\}$ such that**

   (a) r **is not a function**



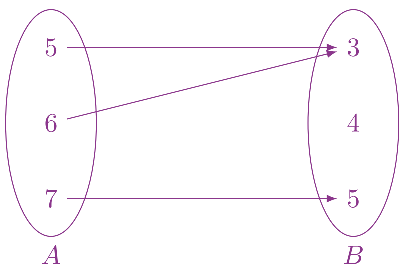
   This is not a function, as one of the points in the domain is not mapped to the range; further, 5 is mapped to two different elements in the range.

   (b) r **is a function from** A **to** B **with the range** $\mathcal{R}(r) = B$



   This relation is a function; every element in the codomain has a corresponding element in the domain, so $\mathcal{R}(r) = B$.

   (c) r **is a function from** A **to** B **with the range** $\mathcal{R}(r) \neq B$



   This relation is a function, but one element in the codomain does not have a corresponding element in the domain, so $\mathcal{R}(r) \neq B$.

6. **Identify the domain and range of each of the following mappings:**

(a) $\left\{(x,y) \in \mathbb{R}^2 \middle| y = \frac{1}{x+1}\right\}$

- Domain: $D = \mathbb{R} - \{-1\}$
- Range: $R = \mathbb{R} - \{0\}$

(b) $\{(x,y) \in \mathbb{N} \times \mathbb{N} | y = x + 5\}$

- Domain: $D = \mathbb{N}$
- Range: $R = \mathbb{N} - \{1, 2, 3, 4, 5\}$

(c) $\left\{(x,y) \in \mathbb{Z} \times \mathbb{Z} \middle| y = \frac{x^2-4}{x-2}\right\}$

- Domain: $D = \mathbb{Z} - \{2\}$
- Range: $R = \mathbb{Z} - \{4\}$

7. **For each of the following sequences, list the first three terms:**

(a) $a_n = \frac{n+1}{2n+3}$

$$\left\{\frac{2}{5}, \quad \frac{3}{7}, \quad \frac{4}{11}, \dots\right\}$$

(b) $b_n = \frac{1}{n!}$

$$\left\{1, \quad \frac{1}{2}, \quad \frac{1}{6}, \dots\right\}$$

(c) $c_n = 1 - 2^{-n}$

$$\left\{\frac{1}{2}, \quad \frac{3}{4}, \quad \frac{7}{8}, \dots\right\}$$

8. **Prove that if $x_n \to L$ and $y_n \to M$, then $x_n + y_n \to L + M$.**

To show: $|(x_n + y_n) - (L + M)| < \varepsilon$

Proof:

Let $\varepsilon > 0$ (by hypothesis)

$\implies \left(\exists N_x \ni n > N_x \Rightarrow |x_n - L| < \frac{\varepsilon}{2}\right) \wedge \left(\exists N_y \ni n > N_x \Rightarrow |y_n - M| < \frac{\varepsilon}{2}\right)$ (by def. of convergence)

Let $n > \max\{N_x, N_y\}$ (defining n)

Consider $\big|(x_n + y_n) - (L + M)\big|$ (summing the sequences & limits)

$= |(x_n - L) + (y_n - M)|$ (by associativity)

$\leq |x_n - L| + |y_n - M|$ (by the triangle inequality)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ (by $n > \max\{N_x, N_y\}$)

$= \varepsilon$ (simplifying)

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9. **Prove that if $a_n \to a$ and $a_n \leq b$ for all n, then $a \leq b$.**

Proof by contradiction to show: $a_n \leq b \; \forall \; n$ and $\exists a_n > b$

Proof:

Let $(a_n \to a) \wedge (a_n \leq b \; \forall \; n)$	(by hypothesis)
Suppose $a > b$	(towards a contradiction)
Let $\varepsilon = a - b$	(defining ε)
$\implies \exists N \ni n > N \implies \lvert a_n - a \rvert < \varepsilon$	(by def. of convergence)
Consider $\lvert a_n - a \rvert < \varepsilon$	
$\implies -\varepsilon < a_n - a < \varepsilon$	(by the absolute value)
$\implies a - \varepsilon < a_n < a + \varepsilon$	(rearranging)
$\implies a - (a - b) < a_n$	(by def. of ε)
$\implies b < a_n$	(simplifying)
Buy $a_n \leq b$ by assumption; thus, a contradiction	
$\implies a \leq b$	■

10. **Consider the following intervals in $\mathbb{R}$. For each, determine if it is closed. If so, give a proof:**

(a) $(-\infty, b]$

This interval is closed. <u>To show</u>: $B_\varepsilon(x) \subseteq (b, \infty)$.
<u>Proof</u>:

Let $[x \in (b, \infty)] \wedge [\varepsilon = x - b]$	(by hypothesis)
Consider $B_\varepsilon(x)$	(defining an ε-ball)
Let $y \in B_\varepsilon(x)$	(picking a point in $B_\varepsilon(x)$)
$\implies x - \varepsilon < y < x + \varepsilon$	(by def. of $B_\varepsilon(x)$)
$\implies x - (x - b) < y < x + (x - b)$	(by def. of ε)
$\implies b < y < 2x + b$	(simplifying)
$\implies b < y < \infty$	($2x + b$ is finite)
$\implies y \in (b, \infty)$	(by def. of (b, ∞))
$\implies B_\varepsilon(x) \subseteq (b, \infty)$	(by def. of subset)

Because x was an arbitrary point and $B_\varepsilon(x) \subseteq (b, \infty)$, the interval is open. Thus, its complement $(-\infty, b]$ is closed. ■

(b) $(a, b]$

This interval is neither open nor closed. Consider a sequence $a_n = a + \frac{1}{kn}$, where k is a constant large enough such that $a_n \in (a, b]$ for all n. This sequence converges to a, but a is not in the set. Thus, the set is not closed.

(c) $[a, \infty)$

This interval is closed.

- Theorem (T1): weak inequalities are preserved in the limit (see problem 10)

<u>To show</u>: $x \in [a, \infty)$.
<u>Proof</u>:

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence $\ni (x_n \to x) \wedge (x_n \in [a, \infty) \; \forall \; n)$	(by hypothesis)
$\implies x_n \geq a \; \forall \; n$	(by $x_n \in [a, \infty)$)
$\implies x \geq a$	(by T1)

Because the convergent sequence and the limit point were arbitrary, it must be the case that the limit point of every convergent sequence is in $[a, \infty)$. ■

(d) $[a, b)$

As in part (b), this interval is neither open nor closed. Consider a sequence $b_n = b - \frac{1}{kn}$, where k is a constant large enough such that $b_n \in [a, b)$ for all n. This sequence converges to b, but b is not in the set. Thus, the set is not closed.

11. **Consider the following sets. If the set is bounded, provide an M and a x such that $B_M(\mathbf{x})$ contains the set.**

(a) $A = \{x | x \in \mathbb{R} \wedge x^2 \leq 10\}$

This set is bounded above and below by $\sqrt{10}$ and $-\sqrt{10}$, respectively. Thus, let $x = 0$ and $M = 4$. Then $B_M(0)$ contains the entire set.

(b) $B = \left\{x | x \in \mathbb{R} \wedge x + \frac{1}{x} < 5\right\}$

The function is bounded above by $\frac{5+\sqrt{21}}{2}$, but is not bounded below—x can take on any value in the real numbers less than zero.

(c) $C = \{(x, y) | (x, y) \in \mathbb{R}^2_+ \wedge xy < 1\}$

This set is not bounded. No matter how large x gets, there exists a y such that $xy < 1$ (and vice versa).

(d) $D = \{(x, y) | (x, y) \in \mathbb{R} \wedge |x| + |y| \leq 10\}$

Both x and y must fall between -10 and 10. Thus, let $x = 0$ and $M = 11$. Then $B_M(0)$ contains the entire set.

12. **Prove that the following functions are continuous using epsilon-delta proofs.**

(a) $f(x) = x + 3$

To show: $|(x + 3) - (x_0) + 3| < \varepsilon$
Proof:

$$
\begin{aligned}
&\text{Let } \varepsilon > 0 \text{ and } x_0 \in \mathcal{D}(f) && \text{(by hypothesis)} \\
&\text{Let } \delta = \varepsilon && \text{(defining } \delta) \\
&\text{Consider } x \in \mathcal{D}(f) \ni |x - x_0| < \delta \\
&\implies |x - x_0| < \varepsilon && \text{(by def. of } \delta) \\
&\implies |x - x_0 + 3 - 3| < \varepsilon && \text{(adding zero)} \\
&\implies |(x - 3) - (x_0 - 3)| < \varepsilon && \text{(by associativity)}
\end{aligned}
$$

■

This is the basic format of a simple continuity proof: pick an arbitrary ε and an arbitrary point in the domain; pick a specific δ (typically a function of ε); show that if x is within δ of x_0, then $f(x)$ must be within ε of $f(x_0)$.

(b) $g(x) = x^2$

To show: $|x^2 - x_0^2| < \varepsilon$
Proof:

$$
\begin{aligned}
&\text{Let } \varepsilon > 0 \text{ and } x_0 \in \mathcal{D}(g) && \text{(by hypothesis)} \\
&\text{Let } \delta \leq \min\left\{1, \frac{\varepsilon}{2 + 2|x_0|}\right\} && \text{(defining } \delta) \\
&\text{Consider } x \in \mathcal{D}(f) \ni |x - x_0| < \delta
\end{aligned}
$$

$$
\begin{aligned}
&\implies |x - x_0| < \frac{\varepsilon}{2 + 2|x_0|} && \text{(by def. of } \delta) \\
&\implies |x - x_0| < \frac{\varepsilon}{1 + \delta + 2|x_0|} && \text{(by } \delta < 1) \\
&\implies |x - x_0| < \frac{\varepsilon}{1 + |x - x_0| + 2|x_0|} && \text{(by } |x - x_0| < \delta) \\
&\implies |x - x_0| < \frac{\varepsilon}{1 + |x - x_0 + 2x_0|} && \text{(by the triangle inequality)} \\
&\implies |x - x_0| < \frac{\varepsilon}{1 + |x + x_0|} && \text{(simplifying)} \\
&\implies |x - x_0||x + x_0| < \varepsilon && \text{(by } |x + x_0| \geq 0) \\
&\implies |(x - x_0)(x + x_0)| < \varepsilon && \text{(by } |ab| = |a||b|) \\
&\implies |x^2 - x_0^2| < \varepsilon && \text{(simplifying)}
\end{aligned}
$$

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Note that throughout, the denominator is slightly more complicated than seems necessary (e.g., there's always a "$1 + \ldots$" on the bottom of a fraction). This is to handle the case where $x_0 = x = 0$.

(c) $h(x) = |x|$

<u>To show</u>: $||x| - |x_0|| < \varepsilon$
<u>Proof</u>:

$$
\begin{aligned}
&\text{Let } \varepsilon > 0 \text{ and } x_0 \in \mathcal{D}(h) && \text{(by hypothesis)} \\
&\text{Let } \delta = \varepsilon && \text{(defining } \delta) \\
&\text{Consider } x \in \mathcal{D}(f) \ni |x - x_0| < \delta && \\
&\implies |x - x_0| < \varepsilon && \text{(by def. of } \delta) \\
&\implies ||x| - |x_0|| < \varepsilon && \text{(by the reverse triangle ineq.)}
\end{aligned}
$$

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This relies on the "reverse triangle inequality," which is easy to show:

$$
\begin{aligned}
&|y| = |x + y - x| \leq |x| + |y - x| && \text{(by the triangle inequalit)} \\
&\implies |y| - |x| \leq |y - x| && \text{(rearranging)} \\
&|x| = |y + x - y| \leq |y| + |x - y| && \text{(by the triangle inequality)} \\
&\implies |x| - |y| \leq |x - y| && \text{(rearranging)}
\end{aligned}
$$

Noting that $|x - y| = |y - x|$, and $|y| - |x| = -(|x| - |y|)$ we then have two conditions:

$$
\begin{aligned}
&\Big(|x| - |y| \leq |x - y|\Big) \wedge \Big(-(|x| - |y|) \leq |x - y|\Big) && \text{(restating inequalities)} \\
&\implies \big||x| - |y|\big| \leq |x - y| && \text{(combining)}
\end{aligned}
$$